Spectral Geometry

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Aix-Marseille Université, CNRS UMR 7332, CPT, 13288 Marseille France

Abstract

The goal of these lectures is to present some fundamentals of noncommutative geometry looking around its spectral approach. Strongly motivated by physics, in particular by relativity and quantum mechanics, Chamseddine and Connes have defined an action based on spectral considerations, the so-called spectral action.
The idea here is to review the necessary tools which are behind this spectral action to be able to compute it first in the case of Riemannian manifolds (Einstein–Hilbert action). Then, all primary objects defined for manifolds will be generalized to reach the level of noncommutative geometry via spectral triples, with the concrete analysis of the noncommutative torus which is a deformation of the ordinary one.
The basics ingredients such as Dirac operators, heat equation asymptotics, zeta functions, noncommutative residues, pseudodifferential operators, or Dixmier traces will be presented and studied within the framework of operators on Hilbert spaces. These notions are appropriate in noncommutative geometry to tackle the case where the space is swapped with an algebra like for instance the noncommutative torus.

1 Motivations

Let us first expose few motivations from physics to study noncommutative geometry which is by essence a spectral geometry. Of course, precise mathematical definitions and results will be given in other sections.

The notion of spectrum is quite important in physics, for instance in classical mechanics, the Fourier spectrum is essential to understand vibrations or the light spectrum in electromagnetism. The notion of spectral theory is also important in functional analysis, where the spectral theorem tells us that any selfadjoint operator A can be seen as an integral over its spectral measure $A = \int_{a\in \mathrm{Sp}(a)} a\, dP_a$ if $\mathrm{Sp}(A)$ is the spectrum of A. This is of course essential in the axiomatic formulation of quantum mechanics, especially in the Heisenberg picture where the observables are selfadjoint operators.
But this notion is also useful in geometry. In special relativity, we consider fields $\psi(\vec{x})$ for $\vec{x} \in \mathbb{R}^4$ and the electric and magnetic fields E, $B \in \mathrm{Function}(M = \mathbb{R}^4, \mathbb{R}^3)$. Einstein introduced in 1915 the gravitational field and the equation of motion of matter. But a problem appeared: what are the physical meaning of coordinates x^μ and equations of fields? Assume the general covariance of field equation. If $g_{\mu\nu}(x)$ or the tetradfield $e^I_\mu(x)$ is a solution (where I is a local inertial reference frame), then, for any diffeomorphism ϕ of M which is active or passive (i.e. change of coordinates), $e'^I_\nu(x) = \frac{\partial x^\mu}{\partial\phi(x)^\nu}\, e^I_\mu(x)$ is also a solution. As a consequence, when relativity became general, the points disappeared and it remained only fields on fields in the sense that there is no fields on a given space-time. But how to practice geometry without space, given usually by a manifold M? In this later case, the spectral approach, namely the control of eigenvalues of the scalar (or spinorial) Laplacian returns important informations on M and one can address the question if they are sufficient: can one hear the shape of M?

There are two natural points of view on the notion of space: one is based on points (of a manifold), this is the traditional geometrical one. The other is based on algebra and this is the spectral one. So the idea is to use algebra of the dual spectral quantities.
This is of course more in the spirit of quantum mechanics but it remains to know what is a quantum geometry with bosons satisfying the Klein-Gordon equation $(\Box + m^2)\psi(\vec{x}) = s_b(\vec{x})$ and fermions satisfying $(i\partial\!\!\!/ - m)\psi(\vec{x}) = s_f(\vec{x})$ for sources s_b, s_f. Here $\partial\!\!\!/$ can be seen as a square root of $\Box$ and the Dirac operator will play a key role in noncommutative geometry.

In some sense, quantum forces and general relativity drive us to a spectral approach of physics, especially of space-time.

Noncommutative geometry, mainly pioneered by A. Connes (see [24, 30]), is based on a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where the $*$-algebra $\mathcal{A}$ generalizes smooth functions on space-time M (or the coordinates) with pointwise product, $\mathcal{H}$ generalizes the Hilbert space of above quoted spinors ψ and $\mathcal{D}$ is a selfadjoint operator on $\mathcal{H}$ which generalizes $\partial\!\!\!/$ via a connection on a vector bundle over M. The algebra $\mathcal{A}$ also acts, via a representation of $*$-algebra, on $\mathcal{H}$.

Noncommutative geometry treats space-time as quantum physics does for the phase-space since it gives a uncertainty principle: under a certain scale, phase-space points are indistinguishable. Below the scale Λ^{-1}, a certain renormalization is necessary. Given a geometry, the notion of action plays an essential role in physics, for instance, the Einstein–Hilbert action in gravity or the Yang–Mills–Higgs action in particle physics. So here, given the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the appropriate notion of action was introduced by Chamseddine and Connes [10] and defined as $S(\mathcal{D}, \Lambda, f) := \mathrm{Tr}\,\big(f(\mathcal{D}/\Lambda)\big)$ where $\Lambda \in \mathbb{R}^+$ plays the role of a cut-off and f is a positive even function. The asymptotic series in $\Lambda \to \infty$ yields to an effective theory. For instance, this action applied to a noncommutative model of space-time $M \times F$ with a fine structure for fermions encoded in a finite geometry F gives rise from pure gravity to the standard model coupled with gravity [11, 20, 30].

The purpose of these notes is mainly to compute this spectral action on few examples like manifolds and the noncommutative torus.
In section 2, we present standard material on pseudodifferential operators over a compact Riemannian manifold. A description of the behavior of the kernel of a ΨDO near the diagonal is given with the important example of elliptic operators. Then follows the notion of Wodzicki residue and its computation. The main point being to understand why it is a residue.

In section 3, the link with the Dixmier trace is shown. Different subspaces of compact operators are described, and in particular, the ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$. Its definition is on purpose because in renormalization theory, one has to control the logarithmic divergency of the series $\sum_{n=1}^{\infty} n^{-1}$. We will see that this "defect" of convergence of the Riemann zeta function (in the sense that this generates a lot of complications of convergence in physics) is in fact an "advantage" because it is precisely the Dixmier trace and more generally the Wodzicki residue which are the right tools which mimics this zeta function: firstly, this controls the spectral aspects of a manifold and secondly they can be generalized to any spectral triple.

In section 4, we recall the basic definition of a Dirac (or Dirac-like) operator on a compact Riemannian manifold $(M,\, g)$ endowed with a vector bundle E. An example is the (Clifford) bundle $E = \mathcal{C}\ell\, M$ where $\mathcal{C}\ell\, T_x^* M$ is the Clifford algebra for $x \in M$. This leads to the notion of spin structure, spin connection ∇^S and Dirac operator $D\!\!\!\!/ = -ic \circ \nabla^S$ where c is the Clifford multiplication. A special focus is put on the change of metrics g under conformal transformations.

In section 5 is presented the fundamentals of heat kernel theory, namely the Green function of the heat operator $e^{t\Delta}$, $t \in \mathbb{R}^+$. In particular, its expansion as $t \to 0^+$ in terms of coefficients of the elliptic operator Δ, with a method to compute the coefficients of this expansion is explained. The idea being to replace the Laplacian Δ by $\mathcal{D}^2$ later on.

In section 6, a noncommutative integration theory is developed around the notion of spectral triple. This means to understand the notion of differential (or pseudodifferential) operators in this context. Within differential calculus, the link between the one-form and the fluctuations of the given $\mathcal{D}$ is outlined.

Section 7 concerns few actions in physics, such as the Einstein–Hilbert or Yang–Mills actions. The spectral action $\mathrm{Tr}\left(f(\mathcal{D}/\Lambda)\right)$ is justified and the link between its asymptotic expansion in Λ and the heat kernel coefficients is given via noncommutative integrals of powers of $|\mathcal{D}|$.

For each section, we suggest references since this review is by no means original.

Notations:
$\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non negative integers.
On $\mathbb{R}^d$, the volume form is $dx = dx^1 \wedge \cdots \wedge dx^d$.
$\mathbb{S}^d$ is the sphere of radius 1 in dimension d. The induced metric $d\xi = |\sum_{j=1}^d (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_d|$ restricts to the volume form on $\mathbb{S}^{d-1}$.
M is a d-dimensional manifold with metric g.
U, V are open set either in M or in $\mathbb{R}^d$.
We denote by $dvol_g$ the unique volume element such that $dvol_g(\xi_1, \cdots, \xi_d) = 1$ for all positively oriented g-orthonormal basis $\{\xi_1, \cdots, \xi_d\}$ of T_xM for $x \in M$. Thus in a local chart $\sqrt{\det\, g_x}\, |dx| = |dvol_g|$.
When $\alpha \in \mathbb{N}^d$ is a multi-index, $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}$, $\quad |\alpha| := \sum_{i=1}^d \alpha_i$, $\quad \alpha! := \alpha_1 \alpha_2 \cdots \alpha_d$.
For $\xi \in \mathbb{R}^d$, $|\xi| := \left(\sum_{k=1}^d |\xi_k|^2\right)^{1/2}$ is the Euclidean metric.
$\mathcal{H}$ is a separable Hilbert space and $\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}), \mathcal{L}^p(\mathcal{H})$ denote respectively the set of bounded, compact and p-Schatten-class operators, so $\mathcal{L}^1(\mathcal{H})$ are trace-class operators.

2 Wodzicki residue and kernel near the diagonal

The aim of this section is to show that the Wodzicki's residue *WRes* is a trace on the set $\Psi DO(M)$ of classical pseudodifferential operators on a compact manifold M of dimension d.

References for this section: Classical books are [99, 102]. For an orientation more in the spirit of noncommutative geometry since here we follow [86, 87] based on [3, 33], see also the excellent books [49, 82, 83, 104, 105].

2.1 A quick overview on pseudodifferential operators

Definition 2.1. *In the following, $m \in \mathbb{C}$. A symbol $\sigma(x, \xi)$ of order m is a C^∞ function: $(x, \xi) \in U \times \mathbb{R}^d \to \mathbb{C}$ satisfying for any compact $K \subset U$ and any $x \in K$*

(i) $|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{K\alpha\beta}\, (1 + |\xi|)^{\Re(m) - |\beta|}$, for some constant $C_{K\alpha\beta}$.

(ii) We suppose that $\sigma(x, \xi) \simeq \sum_{j \geq 0} \sigma_{m-j}(x, \xi)$ where σ_k is homogeneous of degree k in ξ where $\simeq$ means a controlled asymptotic behavior

$$|\partial_x^\alpha \partial_\xi^\beta \Big(\sigma - \sum_{j<N} \sigma_{m-j}\Big)(x, \xi)| \leq C_{KN\alpha\beta}\, |\xi|^{\Re(m) - N - |\beta|} \text{ for } |\xi| \geq 1.$$

The set of symbols of order m is denoted by $S^m(U \times \mathbb{R}^d)$.

A function $a \in C^\infty(U \times U \times \mathbb{R}^d)$ is an amplitude of order m, if for any compact $K \subset U$ and any $\alpha, \beta, \gamma \in \mathbb{N}^d$ there exists a constant $C_{K\alpha\beta\gamma}$ such that

$$|\partial_x^\alpha \partial_y^\gamma \partial_\xi^\beta \, a(x,y,\xi)| \leq C_{K\alpha\beta\gamma} \, (1+|\xi|)^{\Re(m)-|\beta|}, \quad \forall \, x,y \in K, \xi \in \mathbb{R}^d.$$

The set of amplitudes is written $A^m(U)$.

For $\sigma \in S^m(U \times \mathbb{R}^d)$, we get a continuous operator $\sigma(\cdot, D) : u \in C_c^\infty(U) \to C^\infty(U)$ given by

$$\sigma(\cdot, D)(u)(x) := \sigma(x,D)(u) := \tfrac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sigma(x,\xi)\, \widehat{u}(\xi)\, e^{ix\cdot\xi}\, d\xi \tag{1}$$

where $\widehat{\ }$ means the Fourier transform. This operator $\sigma(\cdot, D)$ will be also denoted by $Op(\sigma)$. For instance, if $\sigma(x,\xi) = \sum_\alpha a_\alpha(x)\, \xi^\alpha$, then $\sigma(x,D) = \sum_\alpha a_\alpha(x)\, D_x^\alpha$ with $D_x := -i\partial_x$. Remark that, by transposition, there is a natural extension of $\sigma(\cdot, D)$ from the set $\mathcal{D}'_c(U)$ of distributions with compact support in U to the set of distributions $\mathcal{D}'(U)$.

By definition, the leading term for $|\alpha| = m$ is the *principal symbol* and the *Schwartz kernel of* $\sigma(x,D)$ is defined by

$$k^{\sigma(x,D)}(x,y) := \tfrac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sigma(x,\xi)\, e^{i(x-y)\cdot\xi}\, d\xi = \check{\sigma}_{\xi \to y}(x, x-y)$$

where $\check{\ }$ is the Fourier inverse in variable ξ. Similarly, if the kernel of the operator $Op(a)$ associated to the amplitude a is

$$k^a(x,y) := \tfrac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(x,y,\xi)\, e^{i(x-y)\cdot\xi}\, d\xi. \tag{2}$$

Definition 2.2. *$P : C_c^\infty(U) \to C^\infty(U)$ (or $\mathcal{D}'(U)$) is said to be smoothing if its kernel is in $C^\infty(U \times U)$ and $\Psi DO^{-\infty}(U)$ will denote the set of smoothing operators.*
For $m \in \mathbb{C}$, the set $\Psi DO^m(U)$ of pseudodifferential operators of order m is the set of P such that $P : C_c^\infty(U) \to C^\infty(U)$, $Pu(x) = \big(\sigma(x,D) + R\big)(u)$ where $\sigma \in S^m(U \times \mathbb{R}^d)$, $R \in \Psi DO^{-\infty}$. σ is called the symbol of P.

It is important to quote that a smoothing operator is a pseudodifferential operator whose amplitude is in $A^m(U)$ for all $m \in \mathbb{R}$: by (2), $a(x,y,\xi) := e^{-i(x-y)\cdot\xi}\, k(x,y)\, \phi(\xi)$ where the function $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfies $\int_{\mathbb{R}^d} \phi(\xi)\, d\xi = (2\pi)^d$. The main obstruction to smoothness is on the diagonal since $k^{\sigma(x,D)}$ is C^∞ outside the diagonal.

Few remarks on the duality between symbols and a subset of pseudodifferential operators:

$$\sigma(x,\xi) \in S^m(U \times \mathbb{R}^d) \longleftrightarrow k_\sigma(x,y) \in \mathcal{D}'(U \times U \times \mathbb{R}^d) \longleftrightarrow A = Op(\sigma) \in \Psi DO^m$$

with the definition $\sigma^A(x,\xi) := e^{-ix\cdot\xi}\, A(x \to e^{ix\cdot\xi})$ where A is properly supported, namely, A and its adjoint map the dual of $C^\infty(U)$ (distributions with compact support) into itself. Moreover,

$$\sigma^A \simeq \sum_\alpha \tfrac{(-i)^\alpha}{\alpha!}\, \partial_\xi^\alpha \partial_y^\alpha\, k_\sigma^A(x,y,\xi)_{|y=x}\,, \qquad k_\sigma^A(x,y) =: \tfrac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}\, k^A(x,y,\xi)\, d\xi\,,$$

where $k^A(x,y,\xi)$ is the amplitude of $k_\sigma^A(x,y)$. Actually, $\sigma^A(x,\xi) = e^{iD_\xi\, D_y}\, k^A(x,y,\xi)_{|y=x}$ and $e^{iD_\xi\, D_y} = 1 + iD_\xi D_y - \frac{1}{2}(D_\xi D_y)^2 + \cdots$. Thus $A = Op(\sigma^A) + R$ where R is a regularizing operator on U.

There are two fundamental points about ΨDO's: they form an algebra and this notion is stable by diffeomorphism justifying its extension to manifolds and then to bundles:

Theorem 2.3. *(i) If $P_1 \in \Psi DO^{m_1}$ and $P_2 \in \Psi DO^{m_2}$, then $P_1P_2 \in \Psi DO^{m_1+m_2}$ with symbol*

$$\sigma^{P_1P_2}(x,\xi) \simeq \sum_{\alpha\in\mathbb{N}^d} \tfrac{(-i)^\alpha}{\alpha!}\, \partial_\xi^\alpha \sigma^{P_1}(x,\xi)\, \partial_x^\alpha \sigma^{P_2}(x,\xi).$$

The principal symbol of P_1P_2 is $\sigma^{P_1P_2}_{m_1+m_2}(x,\xi) = \sigma^{P_1}_{m_1}(x,\xi)\,\sigma^{P_2}_{m_2}(x,\xi)$.

(ii) Let $P \in \Psi DO^m(U)$ and $\phi \in Diff(U,V)$ where V is another open set of $\mathbb{R}^d$. The operator $\phi_*P : f \in C^\infty(V) \to P(f\circ\phi)\circ\phi^{-1}$ satisfies $\phi_*P \in \Psi DO^m(V)$ and its symbol is

$$\sigma^{\phi_*P}(x,\xi) = \sigma^P_m\Big(\phi^{-1}(x),(d\phi)^t\xi\Big) + \sum_{|\alpha|>0} \tfrac{(-i)^\alpha}{\alpha!}\, \phi_\alpha(x,\xi)\, \partial_\xi^\alpha \sigma^P\Big(\phi^{-1}(x),(d\phi)^t\xi\Big)$$

where ϕ_α is a polynomial of degree α in ξ. Moreover, its principal symbol is

$$\sigma^{\phi_*P}_m(x,\xi) = \sigma^P_m\Big(\phi^{-1}(x),(d\phi)^t\xi\Big).$$

In other terms, the principal symbol is covariant by diffeomorphism: ${\sigma^{\phi_*P}}_m = \phi_*\sigma^P_m$.

While the proof of formal expressions is a direct computation, the asymptotic behavior requires some care, see [99, 102].

An interesting remark is in order: $\sigma^P(x,\xi) = e^{-ix\cdot\xi}\,P(x\to e^{ix\cdot\xi})$, thus the dilation $\xi\to t\xi$ with $t>0$ gives $t^{-m}\,e^{-itx\cdot\xi}P\,e^{itx\cdot\xi} = t^{-m}\sigma^P(x,t\xi) \simeq t^{-m}\sum_{j\geq 0}\sigma^P_{m-j}(x,t\xi) = \sigma^P_m(x,\xi)+o(t^{-1})$. Thus, if $P\in\Psi DO^m(U)$ with $m\geq 0$,

$$\sigma^P_m(x,\xi) = \lim_{t\to\infty} t^{-m}\,e^{-ith(x)}\,P\,e^{ith(x)}, \text{ where } h\in C^\infty(U) \text{ is (almost) defined by } dh(x)=\xi.$$

2.2 Case of manifolds

Let M be a (compact) Riemannian manifold of dimension d. Thanks to Theorem 2.3, the following makes sense:

Definition 2.4. *$\Psi DO^m(M)$ is defined as the set of operators $P: C^\infty_c(M)\to C^\infty(M)$ such that*

(i) the kernel $k^P \in C^\infty(M\times M)$ off the diagonal,

(ii) The map : $f\in C_c^\infty\big(\phi(U)\big)\to P(f\circ\phi)\circ\phi^{-1}\in C^\infty\big(\phi(U)\big)$ is in $\Psi DO^m\big(\phi(U)\big)$ for every coordinate chart $(U,\phi:U\to\mathbb{R}^d)$.

Of course, this can be generalized: Given a vector bundle E over M, a linear map $P: \Gamma^\infty_c(M,E)\to\Gamma^\infty(M,E)$ is in $\Psi DO^m(M,E)$ when k^P is smooth off the diagonal, and local expressions are ΨDO's with matrix-valued symbols.

The covariance formula implies that σ^P_m is independent of the chosen local chart so is globally defined on the bundle $T^*M\to M$ and σ^P_m is defined for every $P\in\Psi DO^m$ using overlapping charts and patching with partition of unity.

An important class of pseudodifferential operators are those which are invertible modulo regularizing ones:

Definition 2.5. *$P\in\Psi DO^m(M,E)$ is elliptic if $\sigma^P_m(x,\xi)$ is invertible for all $0\neq\xi\in TM^*_x$.*

This means that $|\sigma^P(x,\xi)|\geq c_1(x)|\xi|^m$ for $|\xi|\geq c_2(x)$, $x\in U$ where c_1,c_2 are strictly positive continuous functions on U. This also means that there exists *a parametrix:*

Lemma 2.6. *The following are equivalent:*

(i) $Op(\sigma) \in \Psi DO^m(U)$ *is elliptic.*

(ii) There exist $\sigma' \in S^{-m}(U \times \mathbb{R}^d)$ *such that* $\sigma \circ \sigma' = 1$ *or* $\sigma' \circ \sigma = 1$.

(iii) $Op(\sigma)\, Op(\sigma') = Op(\sigma')\, Op(\sigma) = 1$ *modulo* $\Psi DO^{-\infty}(U)$.

Thus $Op(\sigma') \in \Psi DO^{-m}(U)$ *is also elliptic.*

Remark that any $P \in \Psi DO(M, E)$ can be extended to a bounded operator on $L^2(M, E)$ when $\Re(m) \leq 0$ but this needs an existing scalar product for given metrics on M and E.

Theorem 2.7. *(see [49]) When* $P \in \Psi DO^{-m}(M, E)$ *is elliptic with* $\Re(m) > 0$*, its spectrum is discrete when* M *is compact.*

We rephrase a previous remark (see [4, Proposition 2.1]): Let E be a vector bundle of rank r over M. If $P \in \Psi DO^{-m}(M, E)$, then for any couple of sections $s \in \Gamma^\infty(M, E)$, $t^* \in \Gamma^\infty(M, E^*)$, the operator $f \in C^\infty(M) \to \langle t^*, P(fs)\rangle \in C^\infty(M)$ is in $\Psi DO^m(M)$. This means that in a local chart (U, ϕ), these operators are $r \times r$ matrices of pseudodifferential operators of order $-m$. The total symbol is in $C^\infty(T^*U) \otimes End(E)$ with $End(E) \simeq M_r(\mathbb{C})$. The principal symbol can be globally defined: $\sigma^P_{-m}(x, \xi) : E_x \to E_x$ for $x \in M$ and $\xi \in T^*_x M$, can be seen as a smooth homomorphism homogeneous of degree $-m$ on all fibers of T^*M. We get the simple formula which could be seen as a definition of the principal symbol

$$\sigma^P_{-m}(x, \xi) = \lim_{t \to \infty} t^{-m} \Big(e^{-ith} \cdot P \cdot e^{ith}\Big)(x) \;\text{ for } x \in M,\, \xi \in T^*_x M \tag{3}$$

where $h \in C^\infty(M)$ is such that $d_x h = \xi$.

2.3 Singularities of the kernel near the diagonal

The question to be solved is to define a homogeneous distribution which is an extension on $\mathbb{R}^d$ of a given homogeneous symbol on $\mathbb{R}^d \backslash \{\,0\,\}$. Such extension is a regularization used for instance by Epstein–Glaser in quantum field theory.

The Schwartz space on $\mathbb{R}^d$ is denoted by $\mathcal{S}$ and the space of tempered distributions by $\mathcal{S}'$.

Definition 2.8. *For* $f_\lambda(\xi) := f(\lambda\xi)$, $\lambda \in \mathbb{R}^*_+$, *define* $\tau \in \mathcal{S}' \to \tau_\lambda$ *by* $\langle \tau_\lambda, f\rangle := \lambda^{-d}\langle \tau, f_{\lambda^{-1}}\rangle$ *for all* $f \in \mathcal{S}$. *A distribution* $\tau \in \mathcal{S}'$ *is homogeneous of order* $m \in \mathbb{C}$ *when* $\tau_\lambda = \lambda^m\, \tau$.

Proposition 2.9. *Let* $\sigma \in C^\infty(\mathbb{R}^d \backslash \{\,0\,\})$ *be a homogeneous symbol of order* $k \in \mathbb{Z}$.
(i) If $k > -d$*, then* σ *defines a homogeneous distribution.*
(ii) If $k = -d$*, there is a unique obstruction to an extension of* σ *given by* $c_\sigma = \int_{\mathbb{S}^{d-1}} \sigma(\xi)\, d\xi$*, namely, one can at best extend* σ *in* $\tau \in \mathcal{S}'$ *such that* $\tau_\lambda = \lambda^{-d}\Big(\tau + c_\sigma \log(\lambda)\, \delta_0\Big)$.

In the following result, we are interested by the behavior near the diagonal of the kernel k^P for $P \in \Psi DO$. For any $\tau \in \mathcal{S}'$, we choose the decomposition as $\tau = \phi \circ \tau + (1 - \phi) \circ \tau$ where $\phi \in C^\infty_c(\mathbb{R}^d)$ and $\phi = 1$ near 0. We can look at the infrared behavior of τ near the origin and its ultraviolet behavior near infinity. Remark first that, since $\phi \circ \tau$ has a compact support, $(\phi \circ \tau)^{\vee} \in \mathcal{S}'$, so the regularity of $\tau^{\vee}$ depends only of its ultraviolet part $\Big((1 - \phi) \circ \tau\Big)^{\vee}$.

Proposition 2.10. *Let $P \in \Psi DO^m(U)$, $m \in \mathbb{Z}$. Then, in local form near the diagonal,*

$$k^P(x,y) = \sum_{-(m+d)\le j\le 0} a_j(x, x-y) - c_P(x)\log|x-y| + \mathcal{O}(1)$$

where $a_j(x,y) \in C^\infty\big(U \times U\backslash\{\,x\,\}\big)$ is homogeneous of order j in y and $c_P(x) \in C^\infty(U)$ with

$$c_P(x) = \tfrac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} \sigma^P_{-d}(x,\xi)\,d\xi. \tag{4}$$

We deduce readily the trace behavior of the amplitude of P:

Theorem 2.11. *Let $P \in \Psi DO^m(M,E)$, $m \in \mathbb{Z}$. Then, for any trivializing local coordinates*

$$tr\Big(k^P(x,y)\Big) = \sum_{j=-(m+d)}^{0} a_j(x, x-y) - c_P(x)\log|x-y| + \mathcal{O}(1),$$

where a_j is homogeneous of degree j in y, c_P is intrinsically locally defined by

$$c_P(x) := \tfrac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} tr\Big(\sigma^P_{-d}(x,\xi)\Big)\,d\xi. \tag{5}$$

Moreover, $c_P(x)|dx|$ is a 1-density over M which is functorial for the diffeomorphisms ϕ:

$$c_{\phi_* P}(x) = \phi_*\Big(c_p(x)\Big). \tag{6}$$

The main difficulty is of course to prove that c_P is well defined.

2.4 Wodzicki residue

The claim is that $\int_M c_P(x)|dx|$ is a residue. For this, we embed everything in $\mathbb{C}$. In the same spirit as in Proposition 2.9, one obtains the following

Lemma 2.12. *Every $\sigma \in C^\infty\Big(\mathbb{R}^d\backslash\{\,0\,\}\Big)$ which is homogeneous of degree $m \in \mathbb{C}\backslash\mathbb{Z}$ can be uniquely extended to a homogeneous distribution.*

Definition 2.13. *Let U be an open set in $\mathbb{R}^d$ and Ω be a domain in $\mathbb{C}$.*
A map $\sigma : \Omega \to S^m(U \times \mathbb{R}^d)$ is said to be holomorphic when
the map $z \in \Omega \to \sigma(z)(x,\xi)$ is analytic for all $x \in U$, $\xi \in \mathbb{R}^d$,
the order $m(z)$ of $\sigma(z)$ is analytic on Ω,
the two bounds of Definition 2.1 (i) and (ii) of the asymptotics $\sigma(z) \simeq \sum_j \sigma_{m(z)-j}(z)$ are locally uniform in z.

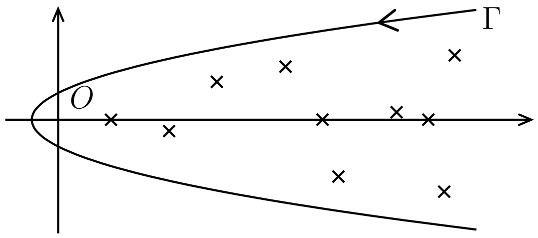
This hypothesis is sufficient to get that the map $z \to \sigma_{m(z)-j}(z)$ is holomorphic from Ω to $C^\infty\Big(U \times \mathbb{R}^d\backslash\{\,0\,\}\Big)$ and the map $\partial_z\sigma(z)(x,\xi)$ is a classical symbol on $U \times \mathbb{R}^d$ such that $\partial_z\sigma(z)(x,\xi) \simeq \sum_{j\ge 0}\partial_z\sigma_{m(z)-j}(z)(x,\xi)$.

Definition 2.14. *The map $P : \Omega \subset \mathbb{C} \to \Psi DO(U)$ is said to be holomorphic if it has the decomposition $P(z) = \sigma(z)(\cdot, D) + R(z)$ (see definition (1)) where $\sigma : \Omega \to S(U \times \mathbb{R}^d)$ and $R : \Omega \to C^\infty(U \times U)$ are holomorphic.*

As a consequence, there exists a holomorphic map from Ω into $\Psi DO(M, E)$ with a holomorphic product (when M is compact).

Elliptic operators: Recall that $P \in \Psi DO^m(U)$, $m \in \mathbb{C}$, is elliptic essentially means that P is invertible modulo smoothing operators. More generally, $P \in \Psi DO^m(M, E)$ is elliptic if its local expression in each coordinate chart is elliptic.

Let $Q \in \Psi DO^m(M, E)$ with $\Re(m) > 0$. We assume that M is compact and Q is elliptic. Thus Q has a discrete spectrum. We assume $\text{Spectrum}(Q) \cap \mathbb{R}^- = \emptyset$ and the existence of a curve Γ whose interior contains the spectrum and avoid branch points of λ^z at $z = 0$:



When $\Re(s) < 0$, $Q^s := \frac{1}{i2\pi} \int_\Gamma \lambda^s \, (\lambda - Q)^{-1} \, d\lambda$ makes sense as operator on $L^2(M, E)$.

The map $s \to Q^s$ is a one-parameter group containing Q^0 and Q^1 which is holomorphic on $\Re(s) \leq 0$. We want to integrate symbols, so we will need the set S_{int} of integrable symbols. Using same type of arguments as in Proposition 2.9 and Lemma 2.12, one proves

Proposition 2.15. *Let $L : \sigma \in S^{\mathbb{Z}}_{int}(\mathbb{R}^d) \to L(\sigma) := \check{\sigma}\,(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sigma(\xi)\, d\xi$. Then L has a unique holomorphic extension $\widetilde{L}$ on $S^{\mathbb{C}\backslash\mathbb{Z}}(\mathbb{R}^d)$. Moreover, when $\sigma(\xi) \simeq \sum_j \sigma_{m-j}(\xi)$, $m \in \mathbb{C}\backslash\mathbb{Z}$, $\widetilde{L}(\sigma) = \big(\sigma - \sum_{j\leq N} \tau_{m-j}\big)^{\vee}(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \big(\sigma - \sum_{j\leq N} \tau_{m-j}\big)(\xi)\, d\xi$ where m is the order of σ, N is an integer with $N > \Re(m) + d$ and τ_{m-j} is the extension of σ_{m-j} of Lemma 2.12.*

Corollary 2.16. *If $\sigma : \mathbb{C} \to S(\mathbb{R}^d)$ is holomorphic and $order\big(\sigma(s)\big) = s$, then $\widetilde{L}\big(\sigma(s)\big)$ is meromorphic with simple poles on $\mathbb{Z}$ and for $p \in \mathbb{Z}$, $\operatorname*{Res}_{s=p} \widetilde{L}\big(\sigma(s)\big) = -\frac{1}{(2\pi)^d} \int_{\mathbb{S}^{d-1}} \sigma_{-d}(p)(\xi)\, d\xi$.*

We are now ready to get the main result of this section which is due to Wodzicki [109, 110].

Definition 2.17. *Let $\mathcal{D} \in \Psi DO(M, E)$ be an elliptic pseudodifferential operator of order 1 on a boundary-less compact manifold M endowed with a vector bundle E.*
Let $\Psi DO_{int}(M, E) := \{\, Q \in \Psi DO^{\mathbb{C}}(M, E) \mid \Re\big(order(Q)\big) < -d \,\}$ be the class of pseudodifferential operators whose symbols are in S_{int}, i.e. integrable in the ξ-variable.
In particular, if $P \in \Psi DO_{int}(M, E)$, then its kernel $k^P(x, x)$ is a smooth density on the diagonal of $M \times M$ with values in $End(E)$.

For $P \in \Psi DO^{\mathbb{Z}}(M, E)$, define

$$WRes\ P := \operatorname*{Res}_{s=0} \operatorname{Tr}\big(P|\mathcal{D}|^{-s}\big). \tag{7}$$

This makes sense because:

Theorem 2.18. *(i) Let $P : \Omega \subset \mathbb{C} \to \Psi DO_{int}(M, E)$ be a holomorphic family. Then the functional map* Tr *: $s \in \Omega \to \operatorname{Tr}(P(s)) \in \mathbb{C}$ has a unique analytic extension on the family $\Omega \to \Psi DO^{\mathbb{C}\backslash\mathbb{Z}}(M, E)$ still denoted by* Tr.

(ii) If $P \in \Psi DO^{\mathbb{Z}}(M,E)$, the map: $s \in \mathbb{C} \to \mathrm{Tr}\left(P|\mathcal{D}|^{-s}\right)$ has at most simple poles on $\mathbb{Z}$ and

$$WRes\ P = -\int_M c_P(x)\,|dx| \tag{8}$$

is independent of $\mathcal{D}$. Recall (see Theorem 2.11) that $c_P(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} tr\Big(\sigma^P_{-d}(x,\xi)\Big)\,d\xi$.

(iii) WRes is a trace on the algebra $\Psi DO^{\mathbb{Z}}(M,E)$.

Proof. The map $s \to \mathrm{Tr}\left(P|D|^{-s}\right)$ is holomorphic on $\mathbb{C}$ and connect $P \in \Psi DO^{\mathbb{C}\backslash\mathbb{Z}}(M,E)$ to the set $\Psi DO_{int}(M,E)$ within $\Psi DO^{\mathbb{C}\backslash\mathbb{Z}}(M,E)$, so a analytic extension of Tr from ΨDO_{int} to $\Psi DO^{\mathbb{C}\backslash\mathbb{Z}}$ is necessarily unique.

(ii) one apply the above machinery:

(1) Notice that Tr is holomorphic on smoothing operator, so, using a partition of unity, we can reduce to a local study of scalar ΨDO's.

(2) First, fix $s = 0$. We are interested in the function $L_\phi(\sigma) := \mathrm{Tr}\left(\phi\,\sigma(x,D)\right)$ with $\sigma \in S_{int}(U\times\mathbb{R}^d)$ and $\phi \in C^\infty(U)$. For instance, if $P = \sigma(\cdot,D)$,

$\mathrm{Tr}(\phi\,P) = \int_U \phi(x)\,k^P(x,x)\,|dx| = \frac{1}{(2\pi)^d}\int_U \phi(x)\,\sigma(x,\xi)\,d\xi\,|dx| = \int_U \phi(x)\,L(\sigma(x,\cdot))\,|dx|$,

so one extends L_ϕ to $S^{\mathbb{C}\backslash\mathbb{Z}}(U\times\mathbb{R}^d)$ with Proposition 2.15 via $\widetilde{L}_\phi(\sigma) = \int_U \phi(x)\,\widetilde{L}_\phi\Big(\sigma(x,\cdot)\Big)\,|dx|$.

(3) If now $\sigma(x,\xi) = \sigma(s)(x,\xi)$ depends holomorphically on s, we get uniform bounds in x, thus we get, via Lemma 2.12 applied to $\widetilde{L}_\phi\Big(\sigma(s)(x,\cdot)\Big)$ uniformly in x, yielding a natural extension to $\widetilde{L}_\phi\Big(\sigma(s)\Big)$ which is holomorphic on $\mathbb{C}\backslash\mathbb{Z}$.

When $\mathrm{order}(\sigma(s)) = s$, the map $\widetilde{L}_\phi\Big(\sigma(s)\Big)$ has at most simple poles on $\mathbb{Z}$ and for each $p \in \mathbb{Z}$, $\operatorname*{Res}_{s=p}\widetilde{L}_\phi\Big(\sigma(s)\Big) = -\frac{1}{(2\pi)^d}\int_U\int_{\mathbb{S}^{d-1}}\phi(x)\,\sigma_{-d}(p)(x,\xi)\,d\xi\,|dx| = -\int_U \phi(x)\,c_{P_p}(x)\,|dx|$ where we used (5) with $P = Op\Big(\sigma_p(x,\xi)\Big)$.

(4) In the general case, we get a unique meromorphic extension of the usual trace Tr on $\Psi DO^{\mathbb{Z}}(M,E)$ that we still denoted by Tr).

When $P:\mathbb{C}\to\Psi DO^{\mathbb{Z}}(M,E)$ is meromorphic with $\mathrm{order}(\Big(P(s)\Big) = s$, then $\mathrm{Tr}\left(P(s)\right)$ has at most poles on $\mathbb{Z}$ and $\operatorname*{Res}_{s=p}\mathrm{Tr}\left(P(s)\right) = -\int_M c_{P(p)}(x)\,|dx|$ for $p\in\mathbb{Z}$. So we get the claim for the family $P(s) := P|\mathcal{D}|^{-s}$.

(iii) Let $P_1, P_2 \in \Psi DO^{\mathbb{Z}}(M,E)$. Since Tr is a trace on $\Psi DO^{\mathbb{C}\backslash\mathbb{Z}}(M,E)$, we get by (i), $\mathrm{Tr}\left(P_1P_2|\mathcal{D}|^{-s}\right) = \mathrm{Tr}\left(P_2|\mathcal{D}|^{-s}P_1\right)$. Moreover

$$WRes\Big(P_1P_2\Big) = \operatorname*{Res}_{s=0}\mathrm{Tr}\left(P_2|\mathcal{D}|^{-s}P_1\right) = \operatorname*{Res}_{s=0}\mathrm{Tr}\left(P_2P_1|\mathcal{D}|^{-s}\right) = WRes\Big(P_2P_1\Big)$$

where for the second equality we used (8) so the residue depends only of the value of $P(s)$ at $s = 0$. □

Note that *WRes* is invariant by diffeomorphism:

$$\text{if } \phi \in \mathrm{Diff}(M),\ WRes(P) = WRes(\phi_* P) \tag{9}$$

which follows from (6). The next result is due to Guillemin and Wodzicki.

Corollary 2.19. *The Wodzicki residue WRes is the only trace (up to multiplication by a constant) on the algebra $\Psi DO^{-m}(M,E)$, $m\in\mathbb{N}$, when M is connected and $d \geq 2$.*

Proof. The restriction to $d \geq 2$ is used only in the part *3)* below. When $d = 1$, T^*M is disconnected and they are two residues.

1) On symbols, derivatives are commutators: $[x^j, \sigma] = i\partial_{\xi_j}\sigma, \quad [\xi_j, \sigma] = -i\partial_{x^j}\sigma.$

2) If $\sigma^P_{-d} = 0$, then $\sigma^P(x,\xi)$ is a finite sum of commutators of symbols:
If $\sigma^P \simeq \sum_j \sigma^P_{m-j}$ with $m = order(P)$, by Euler's theorem, $\sum_{k=1}^d \xi_k\,\partial_{\xi_k}\,\sigma^P_{m-j} = (m-j)\,\sigma^P_{m-j}$ (this is false for $m = j$!) and $\sum_{k=1}^d [x^k,\, \xi_k\,\sigma^P_{m-j}] = i\sum_{k=1}^d \partial_{\xi_k}\xi_k\,\sigma^P_{m-j} = i(m-j+d)\,\sigma^P_{m-j}$. So $\sigma^P = \sum_{k=1}^d [\xi_k\,\tau, x^k]$.

Let T be another trace on $\Psi DO^{\mathbb{Z}}(M,E)$. Then $T(P)$ depends only on σ^P_{-d} because $T([\cdot,\cdot]) = 0$.

3) We have $\int_{\mathbb{S}^{d-1}} \sigma^P_{-d}(x,\xi)\,d|\xi| = 0$ if and only if σ^P_{-d} is sum of derivatives:
The if part is direct (less than more !).
Only if part: σ^P_{-d} is orthogonal to constant functions on the sphere $\mathbb{S}^{d-1}$ and these are kernels of the Laplacian: $\Delta_{\mathbb{S}} f = 0 \iff df = 0 \iff f = cst$. Thus $\Delta_{\mathbb{S}^{d-1}} h = \sigma^P_{-d\restriction \mathbb{S}^{d-1}}$ has a solution h on $\mathbb{S}^{d-1}$. If $\tilde{h}(\xi) := |\xi|^{-d+2}\,h\big(\tfrac{\xi}{|\xi|}\big)$ is its extension to $\mathbb{R}^d\backslash\{0\}$, then we get $\Delta_{\mathbb{R}^d}\tilde{h}(\xi) = |\xi|\sigma^P_{-d}\big(\tfrac{\xi}{|\xi|}\big) = \sigma^P_{-d}(\xi)$ because $\Delta_{\mathbb{R}^d} = r^{1-d}\,\partial_r\big(r^{d-1}\,\partial_r\big) + r^{-2}\,\Delta_{\mathbb{S}^{d-1}}$. This means that $\tilde{h}$ is a symbol of order $d-2$ and $\partial_\xi \tilde{h}$ is a symbol of order $d-1$. As a consequence, $\sigma^P_{-d} = \sum_{k=1}^d \partial^2_{\xi_k}\tilde{h} = -i\sum_{k=1}^d [\partial_{\xi_k}\tilde{h},\, x^k]$ is a sum of commutators.

4) End of proof: the symbol $\sigma^P_{-d}(x,\xi) - \frac{|\xi|^{-d}}{\mathrm{Vol}(\mathbb{S}^{d-1})}\,c_P(x)$ is of order $-d$ with zero integral, thus is a sum of commutators by *3)* and $T(P) = T\big(Op(|\xi|^{-d}\,c_p(x)\big)$, $\forall T \in \Psi DO^{\mathbb{Z}}(M,E)$. So the map $\mu:\ f \in C^\infty_c(U) \to T\big(Op(f|\xi|^{-d})\big)$ is linear, continuous and satisfies $\mu(\partial_{x^k} f) = 0$ because $\partial_{x^k}(f)\,|\xi|^{-d}$ is a commutator if f has a compact support and U is homeomorphic to $\mathbb{R}^d$. As a consequence, μ is a multiple of the Lebesgue integral

$$T(P) = \mu\big(c_P(x)\big) = c\,\textstyle\int_M c_P(x)\,|dx| = c\,WRes(P). \qquad \square$$

Example 2.20. *Laplacian on a manifold M: Let M be a compact Riemannian manifold of dimension d and Δ be the scalar Laplacian which is a differential operator of order 2. Then*

$$WRes\big((1+\Delta)^{-d/2}\big) = Vol\left(\mathbb{S}^{d-1}\right) = \tfrac{2\pi^{d/2}}{\Gamma(d/2)}\,.$$

Proof. $(1+\Delta)^{-d/2} \in \Psi DO(M)$ has order $-d$ and its principal symbol $\sigma^{(1+\Delta)^{-d/2}}_{-d}$ satisfies $\sigma^{(1+\Delta)^{-d/2}}_{-d}(x,\xi) = -\big(g^{ij}_x\,\xi_i\xi_j\big)^{-d/2} = -||\xi||^{-d}_x$. So (8) gives

$$\begin{aligned} WRes\big((1+\Delta)^{-d/2}\big) &= \int_M |dx| \int_{\mathbb{S}^{d-1}} ||\xi||^{-d}_x\,d\xi = \int_M |dx|\sqrt{det\,g_x}\ \mathrm{Vol}\left(\mathbb{S}^{d-1}\right) \\ &= \mathrm{Vol}\left(\mathbb{S}^{d-1}\right)\int_M |dvol_g| = \mathrm{Vol}\left(\mathbb{S}^{d-1}\right). \end{aligned} \qquad \square$$

3 Dixmier trace

References for this section: [33, 49, 68, 87, 104, 105].

The trace Tr on the operators on a Hilbert space $\mathcal{H}$ has an interesting property, it is *normal*. Recall that Tr acting on $\mathcal{B}(\mathcal{H})$ is a particular case of a weight ω acting on a von Neumann algebra $\mathcal{M}$: it is a homogeneous additive map from $\mathcal{M}^+ := \{\, aa^* \mid a \in \mathcal{M}\,\}$ to $[0,\infty]$.

A state is a weight $\omega \in \mathcal{M}^*$ (so $\omega(a) < \infty$, $\forall a \in \mathcal{M}$) such that $\omega(1) = 1$.
A trace is a weight such that $\omega(aa^*) = \omega(a^*a)$ for all $a \in \mathcal{M}$.
A weight ω is normal if $\omega(\sup_\alpha a_\alpha) = \sup_\alpha \omega(a_\alpha)$ whenever $(a_\alpha) \subset \mathcal{M}^+$ is an increasing bounded net. This is equivalent to say that ω is lower semi-continuous with respect to the σ-weak topology.

In particular, *the usual trace* Tr *is normal on* $\mathcal{B}(\mathcal{H})$. Remark that the net $(a_\alpha)_\alpha$ converges in $\mathcal{B}(\mathcal{H})$ and this property looks innocent since a trace preserves positivity.
Nevertheless it is natural to address the question: are all traces (in particular on an arbitrary von Neumann algebra) normal? In 1966, Dixmier answered by the negative [34] by exhibiting non-normal, say singular, traces. Actually, his motivation was to answer the following related question: is any trace ω on $\mathcal{B}(\mathcal{H})$ proportional to the usual trace on the set where ω is finite?

The aim of this section is first to define this Dixmier trace, which essentially means $\mathrm{Tr}_{Dix}(T)$ " = " $\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^{N} \mu_n(T)$, where the $\mu_n(T)$ are the singular values of T ordered in decreasing order and then to relate this to the Wodzicki trace. It is a non-normal trace on some set that we have to identify. Naturally, the reader can feel the link with the Wodzicki trace via Proposition 2.10. We will see that if $P \in \Psi DO^{-d}(M)$ where M is a compact Riemannian manifold of dimension d, then,

$$\mathrm{Tr}_{Dix}(P) = \tfrac{1}{d}\, WRes(P) = \tfrac{1}{d} \int_M \int_{S^*M} \sigma^P_{-d}(x,\xi)\, d\xi |dx|$$

where S^*M is the cosphere bundle on M.

The physical motivation is quite essential: we know how $\sum_{n\in\mathbb{N}^*} \frac{1}{n}$ diverges and this is related to the fact the electromagnetic or Newton gravitational potentials are in $\frac{1}{r}$ which has the same singularity (in one-dimension as previous series). Actually, this (logarithmic-type) divergence appears everywhere in physics and explains the widely use of the Riemann zeta function $\zeta : s \in \mathbb{C} \to \sum_{n\in\mathbb{N}^*} \frac{1}{n^s}$. This is also why we have already seen a logarithmic obstruction in Theorem 2.11 and define a zeta function associated to a pseudodifferential operator P by $\zeta_P(s) = \mathrm{Tr}\left(P|\mathcal{D}|^{-s}\right)$ in (7).

We now have a quick review on the main properties of singular values of an operator.

3.1 Singular values of compact operators

In noncommutative geometry, infinitesimals correspond to compact operators: for $T \in \mathcal{K}(\mathcal{H})$ (compact operators), let $\mu_n(T) := \inf\{\, \|T_{\restriction E^\perp}\| \mid E$ subspace of $\mathcal{H}$ with $\dim(E) = n\,\}$, $n \in \mathbb{N}$. This could looks strange but actually, by mini-max principle, $\mu_n(T)$ is nothing else than the $(n+1)$th of eigenvalues of $|T|$ sorted in decreasing order. Since $\lim_{n\to\infty} \mu_n(T) = 0$, for any $\epsilon > 0$, there exists a finite-dimensional subspace E_ϵ such that $\left\|T_{\restriction E_\epsilon^\perp}\right\| < \epsilon$ and this property being equivalent to T compact, T deserves the name of infinitesimal.

Moreover, we have following properties:

$\mu_n(T) = \mu_n(T^*) = \mu_n(|T|)$.
$T \in \mathcal{L}^1(\mathcal{H})$ (meaning $\|T\|_1 := \mathrm{Tr}(|T|) < \infty$) $\Longleftrightarrow \sum_{n\in\mathbb{N}} \mu_n(T) < \infty$.
$\mu_n(A\,T\,B) \le \|A\|\, \mu_n(T)\, \|B\|$ when $A, B \in \mathcal{B}(\mathcal{H})$.
$\mu_N(U\,T\,U^*) = \mu_N(T)$ when U is a unitary.

Definition 3.1. *For $T \in \mathcal{K}(\mathcal{H})$, the partial trace of order $N \in \mathbb{N}$ is $\sigma_N(T) := \sum_{n=0}^{N} \mu_n(T)$.*

Remark that $\|T\| \leq \sigma_N(T) \leq N\|T\|$ which implies $\sigma_n \simeq \|\cdot\|$ on $\mathcal{K}(\mathcal{H})$. Then

$$\begin{aligned}&\sigma_N(T_1+T_2) \leq \sigma_N(T_1)+\sigma_N(T_2),\\ &\sigma_{N_1}(T_1)+\sigma_{N_2}(T_2) \leq \sigma_{N_1+N_2}(T_1+T_2) \text{ when } T_1, T_2 \geq 0.\end{aligned}$$

This norm σ_N splits: $\sigma_N(T) = \inf\{\, \|x\|_1 + N\,\|y\| \mid T = x+y \text{ with } x \in \mathcal{L}^1(\mathcal{H}),\, y \in \mathcal{K}(\mathcal{H})\,\}$.

This justifies a continuous approach with the

Definition 3.2. *The partial trace of T of order $\lambda \in \mathbb{R}^+$ is*

$$\sigma_\lambda(T) := \inf\{\, \|x\|_1 + \lambda\|y\| \mid T = x+y \textit{ with } x \in \mathcal{L}^1(\mathcal{H}),\, y \in \mathcal{K}(\mathcal{H})\,\}.$$

As before, $\sigma_{\lambda_1}(T_1)+\sigma_{\lambda_2}(T_2) = \sigma_{\lambda_1+\lambda_2}(T_1+T_2)$, for $\lambda_1, \lambda_2 \in \mathbb{R}^+$, $0 \leq T_1, T_2 \in \mathcal{K}(\mathcal{H})$. We define a real interpolate space between $\mathcal{L}^1(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ by

$$\mathcal{L}^{1,\infty} := \{\, T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_{1,\infty} := \sup_{\lambda \geq e} \tfrac{\sigma_\lambda(T)}{\log \lambda} < \infty \,\}.$$

If $\mathcal{L}^p(\mathcal{H})$ is the ideal of operators T such that $\mathrm{Tr}\,\big(|T|^p\big) < \infty$, so $\sigma_\lambda(T) = \mathcal{O}(\lambda^{1-1/p})$, we have

$$\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^{1,\infty} \subset \mathcal{L}^p(\mathcal{H}) \text{ for } p>1, \qquad \|T\| \leq \|T\|_{1,\infty} \leq \|T\|_1\,. \tag{10}$$

Lemma 3.3. *$\mathcal{L}^{1,\infty}$ is a C^*-ideal of $\mathcal{B}(\mathcal{H})$ for the norm $\|\cdot\|_{1,\infty}$. Moreover, it is equal to the Macaev ideal $\mathcal{L}^{1,+} := \{\, T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_{1,+} := \sup_{N \geq 2} \frac{\sigma_N(T)}{\log(N)} < \infty \,\}$.*

3.2 Dixmier trace

We begin with a Cesàro mean of $\frac{\sigma_\rho(T)}{\log \rho}$ with respect of the Haar measure of the group $\mathbb{R}_+^*$:

Definition 3.4. *For $\lambda \geq e$ and $T \in \mathcal{K}(\mathcal{H})$, let $\tau_\lambda(T) := \frac{1}{\log \lambda} \int_e^\lambda \frac{\sigma_\rho(T)}{\log \rho} \frac{d\rho}{\rho}$.*

Clearly, $\sigma_\rho(T) \leq \log \rho\, \|T\|_{1,\infty}$ and $\tau_\lambda(T) \leq \|T\|_{1,\infty}$, thus the map: $\lambda \to \tau_\lambda(T)$ is in $C_b([e,\infty])$. It is not additive on $\mathcal{L}^{1,\infty}$ but this defect is under control:

$$\tau_\lambda(T_1+T_2) - \tau_\lambda(T_1) - \tau_\lambda(T_2) \underset{\lambda\to\infty}{\simeq} \mathcal{O}\Big(\tfrac{\log(\log \lambda)}{\log \lambda}\Big), \text{ when } 0 \leq T_1, T_2 \in \mathcal{L}^{1,\infty} :$$

Lemma 3.5. *In fact, we have*

$$|\,\tau_\lambda(T_1+T_2) - \tau_\lambda(T_1) - \tau_\lambda(T_2)\,| \leq \Big(\tfrac{\log 2(2+\log\log\lambda)}{\log \lambda}\Big)\, \|T_1+T_2\|_{1,\infty}, \textit{ when } T_1, T_2 \in \mathcal{L}^{1,\infty}_+.$$

The Dixmier's idea was to force additivity: since the map $\lambda \to \tau_\lambda(T)$ is in $C_b([e,\infty])$ and $\lambda \to \Big(\frac{\log 2(2+\log\log\lambda)}{\log\lambda}\Big)$ is in $C_0([e,\infty[)$, consider the C^*-algebra $\mathcal{A} := C_b([e,\infty])/C_0([e,\infty[)$. If $[\tau(T)]$ is the class of the map $\lambda \to \tau_\lambda(T)$ in $\mathcal{A}$, previous lemma shows that $[\tau] : T \to [\tau(T)]$ is additive and positive homogeneous from $\mathcal{L}^{1,\infty}_+$ into $\mathcal{A}$ satisfying $[\tau(UTU^*)] = [\tau(T)]$ for any unitary U.

Now let ω be a state on $\mathcal{A}$, namely a positive linear form on $\mathcal{A}$ with $\omega(1) = 1$. Then, $\omega \circ [\tau(\cdot)]$ is a tracial weight on $\mathcal{L}^{1,\infty}_+$. Since $\mathcal{L}^{1,\infty}$ is a C^*-ideal of $\mathcal{B}(\mathcal{H})$, each of its element is generated by (at most) four positive elements, and this map can be extended to a map $\omega \circ [\tau(\cdot)] : T \in \mathcal{L}^{1,\infty} \to \omega([\tau(T)]) \in \mathbb{C}$ such that $\omega([\tau(T_1T_2)]) = \omega([\tau(T_2T_1)])$ for $T_1, T_2 \in \mathcal{L}^{1,\infty}$. This leads to the following definition and result:

Definition 3.6. *The Dixmier trace* Tr_ω *associated to a state* ω *on* $\mathcal{A}$ *is* $\mathrm{Tr}_\omega(\cdot) := \omega \circ [\tau(\cdot)]$.

Theorem 3.7. Tr_ω *is a trace on* $\mathcal{L}^{1,\infty}$ *which depends only on the locally convex topology of* $\mathcal{H}$*, not on its scalar product.*

1) Note that $\mathrm{Tr}_\omega(T) = 0$ if $T \in \mathcal{L}^1(\mathcal{H})$ and all Dixmier traces vanish on the closure for the norm $\|.\|_{1,\infty}$ of the ideal of finite rank operators. So Dixmier traces are not normal.

2) The C^*-algebra $\mathcal{A}$ is not separable, so it is impossible to exhibit any state ω! Despite (10) and the fact that the $\mathcal{L}^p(\mathcal{H})$ are separable ideals for $p \geq 1$, $\mathcal{L}^{1,\infty}$ is not a separable. Moreover, as for Lebesgue integral, there are sets which are not measurable. For instance, a function $f \in C_b([e,\infty])$ has a limit $\ell = \lim_{\lambda\to\infty} f(\lambda)$ if and only if $\ell = \omega(f)$ for all state ω.

Definition 3.8. *The operator* $T \in \mathcal{L}^{1,\infty}$ *is said to be measurable if* $\mathrm{Tr}_\omega(T)$ *is independent of* ω*. In this case,* Tr_ω *is denoted* Tr_{Dix}.

Lemma 3.9. *The operator* $T \in \mathcal{L}^{1,\infty}$ *is measurable and* $\mathrm{Tr}_\omega(T) = \ell$ *if and only if the map* $\lambda \in \mathbb{R}^+ \to \tau_\lambda(T) \in \mathcal{A}$ *converges at infinity to* ℓ.

After Dixmier, singular (i.e. non normal) traces have been deeply investigated, see for instance [72, 74, 75], but we quote only the following characterization of measurability:

If $T \in \mathcal{K}_+(\mathcal{H})$*, then* T *is measurable if and only if* $\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^N \mu_n(T)$ *exists.*

Example 3.10. *Computation of the Dixmier trace of the inverse Laplacian on the torus:*

Let $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ be the d-dimensional torus and $\Delta = -\sum_{i=1}^d \partial_{x^i}^2$ be the scalar Laplacian seen as unbounded operator on $\mathcal{H} = L^2(\mathbb{T}^d)$. We want to compute $\mathrm{Tr}_\omega\Big((1+\Delta)^{-p}\Big)$ for $\frac{d}{2} \leq p \in \mathbb{N}^*$. We use $1+\Delta$ to avoid the kernel problem with the inverse. As the following proof shows, 1 can be replaced by any $\epsilon > 0$ and the result does not depends on ϵ.

Notice that the functions $e_k(x) := \frac{1}{2\pi} e^{ik\cdot x}$ with $x \in \mathbb{T}^d$, $k \in \mathbb{Z}^d = (\mathbb{T}^d)^*$ form a basis of $\mathcal{H}$ of eigenvectors: $\Delta\, e_k = |k|^2\, e_k$. For $t \in \mathbb{R}_+^*$, $e^t\,\mathrm{Tr}\left(e^{-t(1+\Delta)}\right) = \sum_{k\in\mathbb{Z}^d} e^{-t|k|^2} = \left(\sum_{k\in\mathbb{Z}} e^{-tk^2}\right)^d$. We know that $|\int_{-\infty}^\infty e^{-tx^2}\,dx - \sum_{k\in\mathbb{Z}} e^{-tk^2}| \leq 1$, and since the first integral is $\sqrt{\frac{\pi}{t}}$, we get $e^t\,\mathrm{Tr}\left(e^{-t(1+\Delta)}\right) \underset{t\downarrow 0^+}{\simeq} \left(\frac{\pi}{t}\right)^{d/2} =: \alpha\, t^{-d/2}$.

We will use a Tauberian theorem: $\mu_n\Big((1+\Delta)^{-d/2}\Big) \underset{n\to\infty}{\simeq} \Big(\alpha\,\frac{1}{\Gamma(d/2+1)}\Big)\,\frac{1}{n}$, see [49, 54]. Thus

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^N \mu_n\Big((1+\Delta)^{-d/2}\Big) = \frac{\alpha}{\Gamma(d/2+1)} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}.$$

So $(1+\Delta)^{-d/2}$ is measurable and $\mathrm{Tr}_{Dix}\Big((1+\Delta)^{-d/2}\Big) = \mathrm{Tr}_\omega\Big((1+\Delta)^{-d/2}\Big) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$. Since $(1+\Delta)^{-p}$ is traceable for $p > \frac{d}{2}$, $\mathrm{Tr}_{Dix}\Big((1+\Delta)^{-p}\Big) = 0$.

This result has been generalized in Connes' trace theorem [23]: since *WRes* and Tr_{Dix} are traces on $\Psi DO^{-m}(M,E)$, $m \in \mathbb{N}$, we get the following

Theorem 3.11. *Let* M *be a d-dimensional compact Riemannian manifold,* E *a vector bundle over* M *and* $P \in \Psi DO^{-d}(M,E)$*. Then,* $P \in \mathcal{L}^{1,\infty}$*, is measurable and* $\mathrm{Tr}_{Dix}(P) = \frac{1}{d}\, WRes(P)$.

4 Dirac operator

There are several ways to define a Dirac-like operator. The best one is to define Clifford algebras, their representations, the notion of Clifford modules, spin^c structures on orientable manifolds M defined by Morita equivalence between the C^*-algebras $C(M)$ and $\Gamma(\mathcal{C}\ell\, M)$. Then the notion of spin structure and finally, with the notion of spin and Clifford connection, we reach the definition of a (generalized) Dirac operator.
Here we try to bypass this approach to save time.

References: a classical book is [70], but I suggest [46]. Here we follow [86], see also [49].

4.1 Definition and main properties

Let $(M,\, g)$ be a compact Riemannian manifold with metric g, of dimension d and E be a vector bundle over M. An example is the (Clifford) bundle $E = \mathcal{C}\ell\, T^*M$ where the fiber $\mathcal{C}\ell\, T^*_xM$ is the Clifford algebra of the real vector space T^*_xM for $x \in M$ endowed with the nondegenerate quadratic form g.

Given a connection ∇ on E, a differential operator P of order $m \in \mathbb{N}$ on E is an element of $\text{Diff}^m(M,E) := \Gamma\Big(M, End(E)\Big) \cdot Vect\{\, \nabla_{X_1} \cdots \nabla_{X_j} \mid X_j \in \Gamma(M,TM),\, j \leq m\,\}$.
In particular, $\text{Diff}^m(M,E)$ is a subalgebra of $End\Big(\Gamma(M,E)\Big)$ and the operator P has a principal symbol σ_m^P in $\Gamma\Big(T^*M, \pi^*End(E)\Big)$ where $\pi : T^*M \to M$ is the canonical submersion and $\sigma_m^P(x,\xi)$ is given by (3).
Example: Let $E = \bigwedge T^*M$. The exterior product and the contraction given on $\omega, \omega_j \in E$ by $\epsilon(\omega_1)\,\omega_2 := \omega_1 \wedge \omega_2, \qquad \iota(\omega)\,(\omega_1 \wedge \cdots \wedge \omega_m) := \sum_{j=1}^m (-1)^{j-1} g(\omega,\omega_j)\, \omega_1 \wedge \cdots \wedge \widehat{\omega_j} \wedge \cdots \wedge \omega_m$ suggest the following definition $c(\omega) := \epsilon(\omega) + \iota(\omega)$ and one checks that

$$c(\omega_1)\, c(\omega_2) + c(\omega_2)c(\omega_1) = 2g(\omega_1,\omega_2)\, \mathrm{id}_E. \tag{11}$$

E has a natural scalar product: if $e_1, \cdots, e_d$ is an orthonormal basis of T^*_xM, then the scalar product is chosen such that $e_{i_1} \wedge \cdots \wedge e_{i_p}$ for $i_1 < \cdots < i_p$ is an orthonormal basis.
If $d \in \text{Diff}^1$ is the exterior derivative and d^* is its adjoint for the deduced scalar product on $\Gamma(M,E)$, then their principal symbols are

$$\sigma_1^d(\omega) = i\epsilon(\omega), \qquad \sigma_1^{d^*}(\omega) = -i\iota(\omega). \tag{12}$$

This follows from $\sigma_1^d(x,\xi) = \lim_{t\to\infty} \frac{1}{t}\Big(e^{-ith(x)} d e^{ith(x}\Big)(x) = lim_{t\to\infty} \frac{1}{t}\, it\, d_xh = i\, d_xh = i\,\xi$ where h is such that $d_xh = \xi$, so $\sigma_1^d(x,\xi) = i\,\xi$ and similarly for $\sigma_1^{d^*}$.

More generally, if $P \in \text{Diff}^m(M)$, then $\sigma_m^P(dh) = \frac{1}{i^m m!}\,(ad\, h)^m(P)$ with $ad\, h = [\cdot, h]$ and $\sigma_m^{P^*}(\omega) = \sigma_m^P(\omega)^*$ where the adjoint P^* is for the scalar product on $\Gamma(M,E)$ associated to an hermitean metric on E: $\langle \psi, \psi' \rangle := \int_M \langle \psi(x), \psi'(x) \rangle_x\, |dx|$ is a scalar product on $\Gamma(M,E)$.

Definition 4.1. *The operator $P \in Diff^2(M,E)$ is called a generalized Laplacian when its symbol satisfies $\sigma_2^P(x,\xi) = |\xi|_x^2\; id_{E_x}$ for $x \in M$, $\xi \in T^*_xM$.*

This is equivalent to say that, in local coordinates, $P = -\sum_{i,j} g^{ij}(x)\partial_{x^i}\partial_{x^j} + b^j(x)\partial_{x^j} + c(x)$ where the b^j are smooth and c is in $\Gamma\Big(M, End(E)\Big)$.

Definition 4.2. *Assume that $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded vector bundle. When $D \in \mathit{Diff}^{\,1}(M,E)$ and $D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$ (D is odd) where $D^\pm : \Gamma(M,E^\mp) \to \Gamma(M,E^\pm)$, D is called a Dirac operator if $D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix}$ is a generalized Laplacian.*

A good example is given by $E = \bigwedge T^*M = \bigwedge^{even} T^*M \oplus \bigwedge^{odd} T^*M$ and the de Rham operator $D := d + d^*$. It is a Dirac operator since $D^2 = dd^* + d^*d$ is a generalized Laplacian according to (12). D^2 is also called the Laplace–Beltrami operator.

Definition 4.3. *Define $\mathcal{C}\ell\, M$ as the vector bundle over M whose fiber in $x \in M$ is the Clifford algebra $\mathcal{C}\ell\, T^*_xM$ (or $\mathcal{C}\ell\, T_xM$ using the musical isomorphism $X \in TM \leftrightarrow X^\flat \in T^*M$).*

A bundle E is called a Clifford bundle over M when there exists a $\mathbb{Z}_2$-graduate action $c : \Gamma(M, \mathcal{C}\ell\, M) \to End\big(\Gamma(M,E)\big)$.

The main idea which drives this definition is that Clifford actions correspond to principal symbols of Dirac operators:

Proposition 4.4. *If E is a Clifford module, every odd $D \in \mathit{Diff}^{\,1}$ such that $[D,\, f] = i\, c(df)$ for $f \in C^\infty(M)$ is a Dirac operator.*
Conversely, if D is a Dirac operator, there exists a Clifford action c with $c(df) = -i\,[D,\, f]$.

Consider previous example: $E = \bigwedge T^*M = \bigwedge^{even} T^*M \oplus \bigwedge^{odd} T^*M$ is a Clifford module for $c := i(\epsilon + \iota)$ coming from the Dirac operator $D = d + d^*$: by (12)

$$i[D,f] = i[d + d^*,\, f] = i\Big(i\sigma_1^d(df) - i\sigma_1^{d^*}(df)\Big) = -i(\epsilon + \iota)(df).$$

Definition 4.5. *Let E be a Clifford module over M. A connection ∇ on E is a Clifford connection if for $a \in \Gamma(M, \mathcal{C}\ell\, M)$ and $X \in \Gamma(M,TM)$, $[\nabla_X, c(a)] = c(\nabla^{LC}_X\, a)$ where ∇^{LC}_X is the Levi-Civita connection after its extension to the bundle $\mathcal{C}\ell\, M$ (here $\mathcal{C}\ell\, M$ is the bundle with fiber $\mathcal{C}\ell\, T_xM$). A Dirac operator D_∇ is associated to a Clifford connection ∇:*

$$D_\nabla := -i\, c \circ \nabla, \qquad \Gamma(M,E) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes E) \xrightarrow{c\otimes 1} \Gamma(M,E),$$

where we use c for $c \otimes 1$.

Thus if in local coordinates, $\nabla = \sum_{j=1}^d dx^j \otimes \nabla_{\partial_j}$, the associated Dirac operator is given by $D_\nabla = -i \sum_j c(dx^j)\, \nabla_{\partial_i}$. In particular, for $f \in C^\infty(M)$,

$$[D_\nabla,\, f\, id_E] = -i \textstyle\sum_{i=1}^d c(dx^i)\,[\nabla_{\partial_j},\, f] = \sum_{j=1}^d -i\, c(dx^j)\, \partial_j f = -ic(df).$$

By Proposition 4.4, D_∇ deserves the name of Dirac operator!

Examples:

1) For the previous example $E = \bigwedge T^*M$, the Levi-Civita connection is indeed a Clifford connection whose associated Dirac operator coincides with the de Rham operator $D = d + d^*$.

2) *The spinor bundle*: Recall that the spin group Spin_d is the non-trivial two-fold covering of SO_d, so we have $0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_d \xrightarrow{\xi} SO_d \longrightarrow 1$.

Let $\mathrm{SO}(TM) \to M$ be the SO_d-principal bundle of positively oriented orthonormal frames on TM of an oriented Riemannian manifold M of dimension d.

A *spin structure* on an oriented d-dimensional Riemannian manifold (M,g) is a Spin_d-principal bundle $\mathrm{Spin}(TM) \xrightarrow{\pi} M$ with a two-fold covering map $\mathrm{Spin}(TM) \xrightarrow{\eta} \mathrm{SO}(TM)$ such that the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Spin}(TM) \times \mathrm{Spin}_d \arrow[r] \arrow[d, "\eta\times\xi"'] & \mathrm{Spin}(TM) \arrow[d, "\eta"'] \arrow[dr, "\pi"] & \\
\mathrm{SO}(TM) \times \mathrm{SO}_d \arrow[r] & \mathrm{SO}(TM) \arrow[r, "\pi"'] & M
\end{tikzcd}
$$

where the horizontal maps are the actions of Spin_d and SO_d on the principal fiber bundles $\mathrm{Spin}(TM)$ and $\mathrm{SO}(TM)$.

A *spin manifold* is an oriented Riemannian manifold admitting a spin structure.
In above definition, one can replace Spin_d by the group Spin^c_d which is a central extension of SO_d by $\mathbb{T}$: $0 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Spin}^c_d \xrightarrow{\xi} SO_d \longrightarrow 1$.

An oriented Riemannian manifold (M, g) is spin if and only if the second Stiefel–Whitney class of its tangent bundle vanishes. Thus a manifold is spin if and only both its first and second Stiefel–Whitney classes vanish. In this case, the set of spin structures on (M, g) stands in one-to-one correspondence with $H^1(M, \mathbb{Z}_2)$. In particular the existence of a spin structure does not depend on the metric or the orientation of a given manifold.

Let ρ be an irreducible representation of $\mathcal{C}\ell\,\mathbb{C}^d \to End_{\mathbb{C}}(\Sigma_d)$ with $\Sigma_d \simeq \mathbb{C}^{2^{\lfloor d/2 \rfloor}}$ as set of complex spinors. Of course, $\mathcal{C}\ell\,\mathbb{C}^d$ is endowed with its canonical complex bilinear form.

The *spinor bundle* S of M is the complex vector bundle associated to the principal bundle $\mathrm{Spin}(TM)$ with the spinor representation, namely $S := \mathrm{Spin}(TM) \times_{\rho_d} \Sigma_d$. Here ρ_d is a representation of Spin_d on $\mathrm{Aut}(\Sigma_d)$ which is the restriction of ρ.

More precisely, if $d = 2m$ is even, $\rho_d = \rho^+ + \rho^-$ where $\rho^\pm$ are two nonequivalent irreducible complex representations of Spin_{2m} and $\Sigma_{2m} = \Sigma^+_{2m} \oplus \Sigma^-_{2m}$, while for $d = 2m + 1$ odd, the spinor representation ρ_d is irreducible.
In practice, M is a spin manifold means that there exists a Clifford bundle $S = S^+ \oplus S^-$ such that $S \simeq \bigwedge T^*M$. Due to the dimension of M, the Clifford bundle has fiber

$$
\mathcal{C}\ell_x M = \begin{cases} M_{2^m}(\mathbb{C}) \text{ when } d = 2m \text{ is even,} \\ M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) \text{ when } d = 2m + 1. \end{cases}
$$

Locally, the spinor bundle satisfies $S \simeq M \times \mathbb{C}^{d/2}$.
A spin connection $\nabla^S : \Gamma^\infty(M, S) \to \Gamma^\infty(M, S) \otimes \Gamma^\infty(M, T^*M)$ is any connection which is compatible with Clifford action: $[\nabla^S, c(\cdot)] = c(\nabla^{LC} \cdot)$. It is uniquely determined by the choice of a spin structure on M (once an orientation of M is chosen).

Definition 4.6. *<u>The</u> Dirac (called Atiyah–Singer) operator given by the spin structure is*

$$
\not{D} := -i\, c \circ \nabla^S. \tag{13}
$$

In coordinates, $\not{D} = -ic(dx^j)\big(\partial_j - \omega_j(x)\big)$ where ω_j is the spin connection part which can be computed in the coordinate basis $\omega_j = \frac{1}{4}\Big(\Gamma^k_{ji}\, g_{kl} - \partial_i(h^\alpha_j)\delta_{\alpha\beta} h^\beta_l\Big) c(dx^i)\, c(dx^l)$ and the matrix $H := [h^\alpha_j]$ is such that $H^t H = [g_{ij}]$ (we use Latin letters for coordinate basis indices and Greek letters for orthonormal basis indices).

This gives $\sigma^D_1(x, \xi) = c(\xi) + ic(dx^j)\, \omega_j(x)$. Thus in *normal coordinates around* x_0,

$$
c(dx^j)(x_0) = \gamma^j, \qquad \sigma^D_1(x_0, \xi) = c(\xi) = \gamma^j \xi_j
$$

where the γ's are constant hermitean matrices.

The Hilbert space of spinors is

$$
\mathcal{H} = L^2\big((M, g), S\big) := \{\, \psi \in \Gamma^\infty(M, S) \mid \int_M \langle \psi, \psi \rangle_x \, dvol_g(x) < \infty \,\} \tag{14}
$$

where we have a scalar product which is $C^\infty(M)$-valued. On its domain $\Gamma^\infty(M,S)$, $\not{D}$ is symmetric: $\langle \psi, \not{D}\phi\rangle = \langle \not{D}\psi, \phi\rangle$. Moreover, it has a selfadjoint closure (which is $\not{D}^{**}$):

Theorem 4.7. *(see [49, 70, 104, 105]) Let (M,g) be an oriented compact Riemannian spin manifold without boundary. By extension to $\mathcal{H}$, $\not{D}$ is essentially selfadjoint on its original domain $\Gamma^\infty(M,S)$. It is a differential (unbounded) operator of order one which is elliptic.*

There is a nice formula which relates the Dirac operator $\not{D}$ to the spinor Laplacian
$$\Delta^S := -\operatorname{Tr}_g(\nabla^S \circ \nabla^S) : \Gamma^\infty(M,S) \to \Gamma^\infty(M,S).$$
Before to give it, we need to fix few notations: let $R \in \Gamma^\infty\Big(M, \bigwedge^2 T^*M \otimes End(TM)\Big)$ be the *Riemann curvature tensor* with components $R_{ijkl} := g(\partial_i, R(\partial_k,\partial_l)\partial_j)$, the *Ricci tensor* components are $R_{jl} := g^{ik}T_{ijkl}$ and the *scalar curvature* is $s := g^{jl}R_{jl}$.

Proposition 4.8. *Schrödinger–Lichnerowicz formula: if s is the scalar curvature of M, $\not{D}^2 = \Delta^S + \frac{1}{4}s$.*

The proof is just a lengthy computation (see for instance [49]).
We already know via Theorems 2.7 and 4.7 that $\not{D}^{-1}$ is compact so has a discrete spectrum. For $T \in \mathcal{K}_+(\mathcal{H})$, we denote by $\{\lambda_n(T)\}_{n\in\mathbb{N}}$ its spectrum sorted in decreasing order including multiplicity (and in increasing order for an unbounded positive operator T such that T^{-1} is compact) and by $N_T(\lambda) := \#\{\, \lambda_n(T) \mid \lambda_n \leq \lambda \,\}$ its counting function.

Theorem 4.9. *With same hypothesis, the asymptotics of the Dirac operator counting function is $N_{|\not{D}|}(\lambda) \underset{\lambda\to\infty}{\sim} \frac{2^d\, Vol(\mathbb{S}^{d-1})}{d(2\pi)^d}\, Vol(M)\, \lambda^d$ where $Vol(M) = \int_M dvol$.*

We already encounter such computation in Example 3.10.

4.2 Dirac operators and change of metrics

Recall that the spinor bundle S_g and square integrable spinors $\mathcal{H}_g$ defined in (14) depends on the chosen metric g, so we note M_g instead of M and $\mathcal{H}_g := L^2(M_g, S_g)$ and a natural question is: what happens to a Dirac operator when the metric changes?

Let g' be another Riemannian metric on M. Since the space of d-forms is one-dimensional, there exists a positive function $f_{g,g'} : M \to \mathbb{R}^+$ such that $dvol_{g'} = f_{g',g}\, dvol_g$.

Let $I_{g,g'}(x) : S_g \to S_{g'}$ the natural injection on the spinors spaces above point $x \in M$ which is a pointwise linear isometry: $|I_{g,g'}(x)\,\psi(x)|_{g'} = |\psi(x)|_g$. Let us first see its construction: there always exists a g-symmetric automorphism $H_{g,g'}$ of the $2^{\lfloor d/2\rfloor}$- dimensional vector space TM such that $g'(X,Y) = g(H_{g,g'}X, Y)$ for $X,Y \in TM$ so define $\iota_{g,g'}\, X := H_{g,g'}^{-1/2}\, X$. Note that $\iota_{g,g'}$ commutes with right action of the orthogonal group O_d and can be lifted up to a diffeomorphism Pin_d-equivariant on the spin structures associated to g and g' and this lift is denoted by $I_{g,g'}$ (see [6]). This isometry is extended as operator on the Hilbert spaces $I_{g,g'} : \mathcal{H}_g \to \mathcal{H}_{g'}$ with $(I_{g,g'}\,\psi)(x) := I_{g,g'}(x)\,\psi(x)$.

Now define
$$U_{g',g} := \sqrt{f_{g,g'}}\, I_{g',g} \;: \mathcal{H}_{g'} \to \mathcal{H}_g\,.$$
Then by construction, $U_{g',g}$ is a unitary operator from $\mathcal{H}_{g'}$ onto $\mathcal{H}_g$: for $\psi' \in \mathcal{H}_{g'}$,
$$\begin{aligned}\langle U_{g',g}\psi', U_{g',g}\psi'\rangle_{\mathcal{H}_g} &= \int_M |U_{g,g'}\psi'|_g^2\, dvol_g = \int_M |I_{g',g}\psi'|_g^2\, f_{g',g}\, dvol_g = \int_M |\psi'|_{g'}^2\, dvol_{g'}\\ &= \langle \psi_{g'}, \psi_{g'}\rangle_{\mathcal{H}_{g'}}.\end{aligned}$$

So we can realize $\not{D}_{g'}$ as an operator $D_{g'}$ acting on $\mathcal{H}_g$ with

$$D_{g'} : \mathcal{H}_g \to \mathcal{H}_g, \quad D_{g'} := U_{g,g'}^{-1} \not{D}_{g'} U_{g,g'}. \tag{15}$$

This is an unbounded operator on $\mathcal{H}_g$ which has the same eigenvalues as $\not{D}_{g'}$.

In the same vein, the k-th Sobolev space $H^k(M_g, S_g)$ (which is the completion of the space $\Gamma^\infty(M_g, S_g)$ under the norm $\|\psi\|_k^2 = \sum_{j=0}^k \int_M |\nabla^j \psi(x)|^2 \, dx$; be careful, ∇ applied to $\nabla\psi$ is the tensor product connection on $T^*M_g \otimes S_g$ etc, see Theorem 2.7) can be transported: the map $U_{g,g'} : H^k(M_g, S_g) \to H^k(M_{g'}, S_{g'})$ is an isomorphism, see [96]. In particular, (after the transport map U), the domain of $D_{g'}$ and $\not{D}_{g'}$ are the same.

A nice example of this situation is when g' is in the conformal class of g where we can compute explicitly $\not{D}_{g'}$ and $D_{g'}$ [2, 6, 46, 56].

Theorem 4.10. *Let $g' = e^{2h} g$ with $h \in C^\infty(M, \mathbb{R})$. Then there exists an isometry $I_{g,g'}$ between the spinor bundle S_g and $S_{g'}$ such that for $\psi \in \Gamma^\infty(M, S_g)$,*

$$\begin{aligned}
&\not{D}_{g'} I_{g,g'} \psi = e^{-h} I_{g,g'} \Big(\not{D}_g \psi - i \tfrac{d-1}{2} c_g(grad\ h) \psi \Big), \\
&\not{D}_{g'} = e^{-\frac{d+1}{2} h} I_{g,g'} \not{D}_g I_{g,g'}^{-1} e^{\frac{d-1}{2} h}, \\
&D_{g'} = e^{-h/2} \not{D}_g e^{-h/2}.
\end{aligned}$$

Note that $D_{g'}$ is not a Dirac operator as defined in (13) since its principal symbol has an x-dependence: $\sigma^{D_{g'}}(x, \xi) = e^{-h(x)} c_g(\xi)$. The principal symbols of $\not{D}_{g'}$ and $\not{D}_g$ are related by

$$\sigma_d^{\not{D}_{g'}}(x, \xi) = e^{-h(x)/2} U_{g',g}^{-1}(x) \, \sigma_d^{\not{D}_g}(x, \xi) \, U_{g',g}(x) \, e^{-h(x)/2}, \quad \xi \in T_x^* M.$$

Thus $c_{g'}(\xi) = e^{-h(x)} U_{g',g}^{-1}(x) \, c_g(\xi) \, U_{g',g}(x)$, $\xi \in T_x^* M$. This formula gives a verification of $g'(\xi, \eta) = e^{-2h} g(\xi, \eta)$ using $c_g(\xi) c_g(\eta) + c_g(\eta) c_g(\xi) = 2 g(\xi, \eta) \, \mathrm{id}_{S_g}$.

It is also natural to look at the changes on a Dirac operator when the metric g is modified by a diffeomorphism α which preserves the spin structure. The diffeomorphism α can be lifted to a diffeomorphism O_d-equivariant on the O_d-principal bundle of g-orthonormal frames with $\tilde{\alpha} := H_{\alpha^* g, g}^{-1/2} T\alpha$, and this lift also exists on S_g when α preserves both the orientation and the spin structure. However, the last lift is defined up to a $\mathbb{Z}_2$-action which disappears if α is connected to the identity.

The pull-back $g' := \alpha^* g$ of the metric g is defined by $(\alpha^* g)_x(\xi, \eta) = g_{\alpha(x)}(\alpha_*(\xi), \alpha_* \eta)$, $x \in M$, where α_* is the push-forward map : $T_x M \to T_{\alpha(x)} M$. Of course, the metric g' and g are different but the geodesic distances are the same and one checks that $d_{g'} = \alpha^* d_g$.

The principal symbol of a Dirac operator D is $\sigma_d^D(x, \xi) = c_g(\xi)$ so gives the metric g by (11). This information will be used in the definition of a spectral triple. A commutative spectral triple associated to a manifold generates the so-called Connes' distance which is nothing else but the metric distance; see the remark after (24). The link between $d_{\alpha^* g}$ and d_g is explained by (15), since the unitary induces an automorphism of the C^*-algebra $C^\infty(M)$.

5 Heat kernel expansion

References for this section: [4, 43, 44] and especially [107].

The heat kernel is a Green function of the heat operator $e^{t\Delta}$ (recall that $-\Delta$ is a positive operator) which measures the temperature evolution in a domain whose boundary has a given temperature. For instance, the heat kernel of the Euclidean space $\mathbb{R}^d$ is

$$k_t(x,y) = \tfrac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/4t} \text{ for } x \neq y \tag{16}$$

and it solves the heat equation

$$\partial_t k_t(x,y) = \Delta_x k_t(x,y), \quad \forall t > 0,\, x,y \in \mathbb{R}^d, \quad \text{initial condition: } \lim_{t\downarrow 0} k_t(x,y) = \delta(x-y).$$

Actually, $k_t(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ts^2}\, e^{is(x-y)}\, ds$ when $d = 1$.
Note that for $f \in \mathcal{D}(\mathbb{R}^d)$, we have $\lim_{t\downarrow 0}\int_{\mathbf{R}^d} k_t(x,y)\, f(y)\, dy = f(x)$.
For a connected domain (or manifold with boundary with vector bundle V) U, let λ_n be the eigenvalues for the Dirichlet problem of minus the Laplacian

$$-\Delta\phi = \lambda\psi \text{ in } U, \qquad \psi = 0 \text{ on } \partial U.$$

If $\psi_n \in L^2(U)$ are the normalized eigenfunctions, the inverse Dirichlet Laplacian Δ^{-1} is a selfadjoint compact operator, $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_n \to \infty$.

The interest for the heat kernel is that, if $f(x) = \int_0^\infty dt\, e^{-tx}\,\phi(x)$ is the Laplace transform of ϕ, then $\mathrm{Tr}\big(f(-\Delta)\big) = \int_0^\infty dt\,\phi(t)\ \mathrm{Tr}\big(e^{t\,\Delta}\big)$ (if everything makes sense) is controlled by $\mathrm{Tr}\big(e^{t\Delta}\big) = \int_M dvol(x)\, \mathrm{tr}_{V_x} k_t(x,x)$ since $\mathrm{Tr}\big(e^{t\,\Delta}\big) = \sum_{n=1}^\infty e^{t\,\lambda_n}$ and

$$k_t(x,y) = \langle x, e^{t\,\Delta}\, y\rangle = \textstyle\sum_{n,m=1}^\infty \langle x,\psi_m\rangle\, \langle \psi_m, e^{t\,\Delta}\psi_n\rangle\, \langle \psi_n, y\rangle = \sum_{n,m=1}^\infty \overline{\psi_n(x)}\,\psi_n(y)\, e^{t\,\lambda_n}.$$

So it is useful to know the asymptotics of the heat kernel k_t on the diagonal of $M \times M$ especially near $t = 0$.

5.1 The asymptotics of heat kernel

Let now M be a smooth compact Riemannian manifold without boundary, V be a vector bundle over M and $P \in \Psi DO^m(M,V)$ be a positive elliptic operator of order $m > 0$. If $k_t(x,y)$ is the kernel of the heat operator e^{-tP}, then the following asymptotics exits on the diagonal:

$$k_t(x,x) \underset{t\downarrow 0^+}{\sim} \sum_{k=0}^\infty a_k(x)\, t^{(-d+k)/m}$$

which means that $\Big|k_t(x,x) - \sum_{k\leq k(n)} a_k(x)\, t^{(-d+k)/m}\Big|_{\infty,n} < c_n\, t^n$ for $0 < t < 1$ where we used $|f|_{\infty,n} := \sup_{x\in M}\sum_{|\alpha|\leq n} |\partial_x^\alpha f|$ (since P is elliptic, $k_t(x,y)$ is a smooth function of (t,x,y) for $t > 0$, see [43, section 1.6, 1.7]).

More generally, we will use $k(t,f,P) := \mathrm{Tr}\big(f\, e^{-tP}\big)$ where f is a smooth function. We have similarly

$$k(t,f,P) \underset{t\downarrow 0^+}{\sim} \sum_{k=0}^\infty a_k(f,P)\, t^{(-d+k)/m}. \tag{17}$$

The utility of function f will appear later for the computation of coefficients a_k.
The following points are of importance:

1) The existence of this asymptotics is non-trivial [43, 44].

2) The coefficients $a_{2k}(f,P)$ can be computed locally as integral of local invariants: Recall that a locally computable quantity is the integral on the manifold of a local frame-independent smooth function of one variable, depending only on a finite number of derivatives of a finite number of terms in the asymptotic expansion of the total symbol of P.

In noncommutative geometry, local generally means that it is concentrated at infinity in momentum space.

3) The odd coefficients are zero: $a_{2k+1}(f,P)=0$.

For instance, let us assume from now on that P is a Laplace type operator of the form

$$P = -(g^{\mu\nu}\partial_\mu\partial_\nu + \mathbb{A}^\mu\partial_\mu + \mathbb{B}) \tag{18}$$

where $(g^{\mu\nu})_{1\le\mu,\nu\le d}$ is the inverse matrix associated to the metric g on M, and $\mathbb{A}^\mu$ and $\mathbb{B}$ are smooth $L(V)$-sections on M (endomorphisms) (see also Definition 4.1). Then (see [44, Lemma 1.2.1]) there is a unique connection ∇ on V and a unique endomorphism E such that

$$P = -(\mathrm{Tr}_g \nabla^2 + E), \quad \nabla^2(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{\nabla^{LC}_X Y}\,,$$

X,Y are vector fields on M and ∇^{LC} is the Levi-Civita connection on M. Locally

$$\mathrm{Tr}_g \nabla^2 := g^{\mu\nu}(\nabla_\mu\nabla_\nu - \Gamma^\rho_{\mu\nu}\nabla_\rho)$$

where $\Gamma^\rho_{\mu\nu}$ are the Christoffel coefficients of ∇^{LC}. Moreover (with local frames of T^*M and V), $\nabla = dx^\mu \otimes (\partial_\mu + \omega_\mu)$ and E are related to $g^{\mu\nu}$, $\mathbb{A}^\mu$ and $\mathbb{B}$ through

$$\begin{aligned}\omega_\nu &= \tfrac12 g_{\nu\mu}(\mathbb{A}^\mu + g^{\sigma\varepsilon}\Gamma^\mu_{\sigma\varepsilon}\,\mathrm{id}_V)\,,\\ E &= \mathbb{B} - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma^\sigma_{\nu\mu})\,.\end{aligned}$$

In this case, the coefficients $a_k(f,P) = \int_M dvol_g\, \mathrm{tr}_E\big(f(x)\,a_k(P)(x)\big)$ and the $a_k(P) = c_i\,\alpha^i_k(P)$ are linear combination with constants c_i of all possible independent invariants $\alpha^i_k(P)$ of dimension k constructed from E, Ω, R and their derivatives (Ω is the curvature of the connection ω, and R is the Riemann curvature tensor). As an example, for $k=2$, E and s are the only independent invariants.

Point 3) follow since there is no odd-dimension invariant.

If s is the scalar curvature and ';' denote multiple covariant derivative with respect to Levi-Civita connection on M, one finds, using variational methods

$$\begin{aligned}a_0(f,P) &= (4\pi)^{-d/2}\int_M dvol_g\,\mathrm{tr}_V(f)\,,\\ a_2(f,P) &= \tfrac{(4\pi)^{-d/2}}{6}\int_M dvol_g\,\mathrm{tr}_V\big[(f(6E+s)\big],\\ a_4(f,P) &= \tfrac{(4\pi)^{-d/2}}{360}\int_M dvol_g\,\mathrm{tr}_V\big[f(60E_{;kk} + 60Es + 180E^2 + 12R_{;kk} + 5s^2\\ &\qquad\qquad - 2R_{ij}R_{ij} + 2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij})\big].\end{aligned} \tag{19}$$

The coefficient a_6 was computed by Gilkey, a_8 by Amsterdamski, Berkin and O'Connor and a_{10} in 1998 by van de Ven [108]. Some higher coefficients are known in flat spaces.

5.2 Wodzicki residue and heat expansion

Wodzicki has proved that, in (17), $a_k(P)(x) = \frac1m\, c_{P^{(k-d)/m}}(x)$ is true not only for $k=0$ as seen in Theorem 3.11 (where $P \leftrightarrow P^{-1}$), but for all $k\in\mathbb{N}$. In this section, we will prove this result when P is is the inverse of a Dirac operator and this will be generalized in the next section.

Let M be a compact Riemannian manifold of dimension d even, E a Clifford module over M and D be the Dirac operator (definition 4.2) given by a Clifford connection on E. By Theorem 4.7, D is a selfadjoint (unbounded) operator on $\mathcal{H} := L^2(M,S)$.

We are going to use the heat operator e^{-tD^2} since D^2 is related to the Laplacian via the Schrödinger–Lichnerowicz formula (4.2) and since the asymptotics of the heat kernel of this Laplacian is known.

For $t > 0$, $e^{-tD^2} \in \mathcal{L}^1$: follows from $e^{-tD^2} = (1 + D^2)^{(d+1)/2}\, e^{-tD^2}(1 + D^2)^{-(d+1)/2}$, since $(1+D^2)^{-(d+1)/2} \in \mathcal{L}^1$ and $\lambda \to (1+\lambda^2)^{(d+1)/2}e^{-t\lambda^2}$ is bounded. So $\mathrm{Tr}\left(e^{-tD^2}\right) = \sum_n e^{-t\lambda_n^2} < \infty$.

Moreover, the operator e^{-tD^2} has a smooth kernel since it is regularizing ([70]) and the asymptotics of its kernel is, see (16): $k_t(x,y) \underset{t\downarrow 0^+}{\sim} \frac{1}{(4\pi t)^{d/2}} \sqrt{det\, g_x} \sum_{j\geq 0} k_j(x,y)\, t^j\, e^{-d_g(x,y)^2/4t}$ where k_j is a smooth section on $E^* \otimes E$. Thus

$$\mathrm{Tr}\left(e^{-tD^2}\right) \underset{t\downarrow 0^+}{\sim} \sum_{j\geq 0} t^{(j-d)/2}\, a_j(D^2) \tag{20}$$

with for $j \in \mathbb{N}$, $a_{2j}(D^2) := \frac{1}{(4\pi)^{d/2}} \int_M \mathrm{tr}\Big(k_j(x,x)\Big) \sqrt{det\, g_x}\, |dx|, \quad a_{2j+1}(D^2) = 0.$

Theorem 5.1. *For any integer p, $0 \leq p \leq d$, $D^{-p} \in \Psi DO^{-p}(M,E)$ and*

$$WRes\Big(D^{-p}\Big) = \tfrac{2}{\Gamma(p/2)} a_{d-p}(D^2) = \tfrac{2}{(4\pi)^{d/2}\Gamma(p/2)} \int_M tr\Big(k_{(d-p)/2}(x,x)\Big)\, dvol_g(x).$$

Few remarks are in order:

1) If $p = d$, $WRes\Big(D^{-d}\Big) = \frac{2}{\Gamma(p/2)}\, a_0(D^2) = \frac{2}{\Gamma(p/2)}\, \frac{Rank(E)}{(4\pi)^{d/2}}\, \mathrm{Vol}(M)$.

Since $\mathrm{Tr}(e^{-tD^2}) \underset{t\downarrow 0^+}{\sim} a_0(D^2)\, t^{-d/2}$, the Tauberian theorem used in Example 3.10 implies that $D^{-d} = (D^{-2})^{d/2}$ is measurable and we obtain Connes' trace Theorem 3.11

$$\mathrm{Tr}_{Dix}(D^{-d}) = \mathrm{Tr}_\omega(D^{-d}) = \tfrac{a_0(D^2)}{\Gamma(d/2+1)} = \tfrac{1}{d}\, WRes(D^{-d}).$$

2) When $D = \not{D}$ and E is the spinor bundle, the Seeley-deWit coefficient $a_2(\mathcal{D}^2)$ (see (19) with $f = 1$) can be easily computed (see [43, 49]): if s is the scalar curvature,

$$a_2(\not{D}^2) = -\tfrac{1}{12(4\pi)^{d/2}} \int_M s(x)\, dvol_g(x). \tag{21}$$

So $WRes\Big(\not{D}^{-d+2}\Big) = \frac{2}{\Gamma(d/2-1)}\, a_2(\not{D}^2) = c \int_M s(x)\, dvol_g(x)$. This is a quite important result since this last integral is nothing else but the Einstein–Hilbert action. In dimension 4, this is an example of invariant by diffeomorphisms, see (9).

6 Noncommutative integration

The Wodzicki residue is a trace so can be viewed as an integral. But of course, it is quite natural to relate this integral to zeta functions used in (7): with notations of Section 2.4, let $P \in \Psi DO^{\mathbb{Z}}(M,E)$ and $D \in \Psi DO^1(M,E)$ which is elliptic. The definition of zeta function $\zeta_D^P(s) := \mathrm{Tr}\left(P\, |D|^{-s}\right)$ has been useful to prove that $WRes\, P = \underset{s=0}{\mathrm{Res}}\, \zeta_D^P(s) = -\int_M c_P(x)\, |dx|$.

The aim now is to extend this notion to noncommutative spaces encoded in the notion of spectral triple. References: [24, 30, 33, 36, 49].

6.1 Notion of spectral triple

The main properties of a compact spin Riemannian manifold M can be recaptured using the following triple $(\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M,S), \not{D})$. The coordinates $x = (x^1, \cdots, x^d)$ are exchanged with the algebra $C^\infty(M)$, the Dirac operator $\mathcal{D}$ gives the dimension d as seen in Theorem 4.9, but also the metric of M via Connes formula and more generally generates a quantized calculus. The idea is to forget about the commutativity of the algebra and to impose axioms on a triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to generalize the above one in order to be able to obtain appropriate definitions of other notions: pseudodifferential operators, measure and integration theory, KO-theory, orientability, Poincaré duality, Hochschild (co)homology etc.

Definition 6.1. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is the data of an involutive (unital) algebra $\mathcal{A}$ with a faithful representation π on a Hilbert space $\mathcal{H}$ and a selfadjoint operator $\mathcal{D}$ with compact resolvent (thus with discrete spectrum) such that $[\mathcal{D}, \pi(a)]$ is bounded for any $a \in \mathcal{A}$.*

We could impose the existence of a C^*-algebra A where $\mathcal{A} := \{\, a \in A \mid [\mathcal{D}, \pi(a)] \text{ is bounded} \,\}$ is norm dense in A so $\mathcal{A}$ is a pre-C^*-algebra stable by holomorphic calculus.

When there is no confusion, we will write a instead of $\pi(a)$.
We now give useful definitions:

Definition 6.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple.*

It is even if there is a grading operator χ s.t. $\chi = \chi^*$, $[\chi, \pi(a)] = 0$, $\forall a \in \mathcal{A}, \mathcal{D}\chi = -\chi\mathcal{D}$.

It is real of KO-dimension $d \in \mathbb{Z}/8$ if there is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that $J\mathcal{D} = \epsilon\, \mathcal{D}J$, $J^2 = \epsilon'$, $J\chi = \epsilon''\, \chi J$ with the following table for the signs $\epsilon, \epsilon', \epsilon''$

d	0	1	2	3	4	5	6	7
ϵ	1	-1	1	1	1	-1	1	1
ϵ'	1	1	-1	-1	-1	-1	1	1
ϵ''	1		-1		1		-1	

(22)

and the following commutation rules

$$[\pi(a), \pi(b)^\circ) = 0, \qquad \Big[[\mathcal{D}, \pi(a)], \pi(b)^\circ\Big] = 0, \ \forall a, b \in \mathcal{A} \tag{23}$$

where $\pi(a)^\circ := J\pi(a^*)J^{-1}$ is a representation of the opposite algebra $\mathcal{A}^\circ$.

It is d-summable (or has metric dimension d) if the singular values of $\mathcal{D}$ behave like $\mu_n(\mathcal{D}^{-1}) = \mathcal{O}(n^{-1/d})$.

It is regular if $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ are in the domain of δ^n for all $n \in \mathbb{N}$ where $\delta(T) := [|\mathcal{D}|, T]$.

It satisfies the finiteness condition if the space of smooth vectors $\mathcal{H}^\infty := \bigcap_k \operatorname{Dom} \mathcal{D}^k$ is a finitely projective left $\mathcal{A}$-module.

It satisfies the orientation condition if there is a Hochschild cycle $c \in Z_d(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ such that $\pi_\mathcal{D}(c) = \chi$, where $\pi_\mathcal{D}\big((a \otimes b^\circ) \otimes a_1 \otimes \cdots \otimes a_d\big) := \pi(a)\pi(b)^\circ[\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_d)]$ and d is its metric dimension.

An interesting example of noncommutative space of non-zero KO-dimension is given by the finite part of the noncommutative standard model [20, 27, 30].

Moreover, the reality (or charge conjugation in the commutative case) operator J is related to Tomita theory [101].

A reconstruction of the manifold is possible, starting only with a spectral triple where the algebra is commutative (see [28] for a more precise formulation, and also [92]):

Theorem 6.3. *[28] Given a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfying the above axioms, then there exists a compact spin^c manifold M such that $\mathcal{A} \simeq C^\infty(M)$ and $\mathcal{D}$ is a Dirac operator.*

The manifold is known as a set, $M = \mathrm{Sp}(\mathcal{A}) = \mathrm{Sp}(A)$. Notice that $\mathcal{D}$ is known only via its principal symbol, so is not unique. J encodes the nuance between spin and spin^c structures. The spectral action selects the Levi-Civita connection so *the* Dirac operator $\not{D}$.

The way, the operator $\mathcal{D}$ recaptures the original Riemannian metric g of M is via the Connes' distance: the map

$$d(\phi_1, \phi_2) := \sup\{\, |\phi_1(a) - \phi_2(a)| \mid \|[\mathcal{D}, \pi(a)]\| \leq 1, a \in \mathcal{A}\,\} \tag{24}$$

defines a distance (eventually infinite) between two states ϕ_1, ϕ_2 on the C^*-algebra A.

The role of $\mathcal{D}$ is non only to provide a metric by (24), but its homotopy class represents the K-homology fundamental class of the noncommutative space $\mathcal{A}$.

It is known that one cannot hear the shape of a drum since the knowledge of the spectrum of a Laplacian does not determine the metric of the manifold, even if its conformal class is given [7]. But Theorem 6.3 shows that one can hear the shape of a spinorial drum (or better say, of a spectral triple) since the knowledge of the spectrum of the Dirac operator and the volume form, via its cohomological content, is sufficient to recapture the metric and spin structure. See however the more precise refinement made in [29].

6.2 Notion of pseudodifferential operators

Definition 6.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple.*
For $t \in \mathbb{R}$ define the map $F_t : T \in \mathcal{B}(\mathcal{H}) \to e^{it|\mathcal{D}|} T e^{-it|\mathcal{D}|}$ and for $\alpha \in \mathbb{R}$

$$OP^0 := \{\, T \mid t \to F_t(T) \in C^\infty\big(\mathbb{R}, \mathcal{B}(\mathcal{H})\big)\,\} \text{ is the set of operators or order} \leq 0,$$
$$OP^\alpha := \{\, T \mid T|\mathcal{D}|^{-\alpha} \in OP^0\,\} \text{ is the set of operators of order} \leq \alpha.$$

Moreover, we set $\delta(T) := [|\mathcal{D}|, T], \quad \nabla(T) := [\mathcal{D}^2, T]$.

For instance, $C^\infty(M) = OP^0 \bigcap L^\infty(M)$ and $L^\infty(M)$ is the von Neumann algebra generated by $\mathcal{A} = C^\infty(M)$. The spaces OP^α have the expected properties:

Proposition 6.5. *Assume that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is regular so $\mathcal{A} \subset OP^0 = \bigcap_{k\geq 0} \mathrm{Dom}\, \delta^k \subset \mathcal{B}(\mathcal{H})$. Then, for any $\alpha, \beta \in \mathbb{R}$,*

$$OP^\alpha\, OP^\beta \subset OP^{\alpha+\beta}, \quad OP^\alpha \subset OP^\beta \text{ if } \alpha \leq \beta, \quad \delta(OP^\alpha) \subset OP^\alpha, \quad \nabla(OP^\alpha) \subset OP^{\alpha+1}.$$

As an example, let us compute the order of $X = a\,|\mathcal{D}|\,[\mathcal{D}, b]\,\mathcal{D}^{-3}$: since the order of a is 0, of $|\mathcal{D}|$ is 1, of $[\mathcal{D}, b]$ is 0 and of $\mathcal{D}^{-3}$ is -3, we get $X \in OP^{-2}$.

Definition 6.6. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple and $\mathcal{D}(\mathcal{A})$ be the polynomial algebra generated by $\mathcal{A}$, $\mathcal{A}^\circ$, $\mathcal{D}$ and $|\mathcal{D}|$. Define the set of pseudodifferential operators as*

$$\Psi(\mathcal{A}) := \{\, T \mid \forall N \in \mathbb{N}, \exists P \in \mathcal{D}(\mathcal{A}), R \in OP^{-N}, p \in \mathbb{N} \text{ such that } T = P|\mathcal{D}|^{-p} + R\,\}$$

The idea behind this definition is that we want to work modulo the set $OP^{-\infty}$ *of smoothing operators*. This explains the presence of the arbitrary N and R. In the commutative case where $\mathcal{D} \in \mathrm{Diff}^1(M, E)$, we get the natural inclusion $\Psi\big(C^\infty(M)\big) \subset \Psi DO(M, E)$.
The reader should be aware that Definition 6.6 is not exactly the same as in [30, 33, 49] since it pays attention to the reality operator J when it is present.

6.3 Zeta-functions and dimension spectrum

Definition 6.7. *For $P \in \Psi^*(\mathcal{A})$, we define the zeta-function associated to P (and $\mathcal{D}$) by*

$$\zeta_{\mathcal{D}}^{P} : s \in \mathbb{C} \to \mathrm{Tr}\left(P|\mathcal{D}|^{-s}\right) \tag{25}$$

which makes sense since for $\Re(s) \gg 1$, $P|\mathcal{D}|^{-s} \in \mathcal{L}^1(\mathcal{H})$.

The dimension spectrum Sd of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is the set $\{$ poles of $\zeta_{\mathcal{D}}^{P}(s) \,|\, P \in \Psi(\mathcal{A}) \cap OP^0 \}$. It is said simple if it contains poles of order at most one.

The noncommutative integral of P is defined by $⨍ P := \mathop{\mathrm{Res}}_{s=0} \zeta_{\mathcal{D}}^{P}(s)$.

In (25), we assume $\mathcal{D}$ invertible since otherwise, one can replace $\mathcal{D}$ by the invertible operator $\mathcal{D} + P$, P being the projection on $\mathrm{Ker}\, \mathcal{D}$. This change does not modify the computation of the integrals $⨍$ which follow since $⨍ X = 0$ when X is a trace-class operator.

6.4 One-forms and fluctuations of $\mathcal{D}$

The unitary group $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$ gives rise to the automorphism $\alpha_u : a \in \mathcal{A} \to uau^* \in \mathcal{A}$. This defines the inner automorphisms group $Inn(\mathcal{A})$ which is a normal subgroup of the automorphisms $Aut(\mathcal{A}) := \{ \alpha \in Aut(A) \,|\, \alpha(\mathcal{A}) \subset \mathcal{A} \}$. For instance, in case of a gauge theory, the algebra $\mathcal{A} = C^\infty\big(M, M_n(\mathbb{C})\big) \simeq C^\infty(M) \otimes M_n(\mathbb{C})$ is typically used. Then, $Inn(\mathcal{A})$ is locally isomorphic to $\mathcal{G} = C^\infty\big(M, PSU(n)\big)$. Since $Aut\big(C^\infty(M)\big) \simeq \mathrm{Diff}(M)$, we get a complete parallel analogy between following two exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Inn(\mathcal{A}) & \longrightarrow & Aut(\mathcal{A}) & \longrightarrow & Aut(\mathcal{A})/Inn(\mathcal{A}) & \longrightarrow & 1, \\ 1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G} \rtimes \mathrm{Diff}(M) & \longrightarrow & \mathrm{Diff}(M) & \longrightarrow & 1. \end{array}$$

Thus the internal symmetries of physics have to be replaced by the inner automorphisms. *The appropriate framework for inner fluctuations of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is Morita equivalence, see [24, 49].*

Definition 6.8. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple.*
One-forms are defined as $\Omega^1_{\mathcal{D}}(\mathcal{A}) := span\{ adb \mid a, b \in \mathcal{A} \}$, $db := [\mathcal{D}, b]$ which is a $\mathcal{A}$-bimodule.

The Morita equivalence which does not change neither the algebra $\mathcal{A}$ nor the Hilbert space $\mathcal{H}$, gives a natural hermitean fluctuation of $\mathcal{D}$: $\mathcal{D} \to \mathcal{D}_A := \mathcal{D} + A$ with $A = A^* \in \Omega^1_{\mathcal{D}}(\mathcal{A})$. For instance, in commutative geometries, $\Omega^1_{\not{D}}\big(C^\infty(M)\big) = \{ c(da) \,|\, a \in C^\infty(M) \}$.

When a reality operator J exists, we also want $\mathcal{D}_A J = \epsilon\, J \mathcal{D}_A$, so we choose

$$\mathcal{D}_{\widetilde{A}} := \mathcal{D} + \widetilde{A}, \quad \widetilde{A} := A + \epsilon J A J^{-1}, \quad A = A^*. \tag{26}$$

The next two results show that, with the same algebra $\mathcal{A}$ and Hilbert space $\mathcal{H}$, a fluctuation of $\mathcal{D}$ still give rise to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_A)$ or $(\mathcal{A}, \mathcal{H}, \mathcal{D}_{\widetilde{A}})$.

Lemma 6.9. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with a reality operator J and chirality χ. If $A \in \Omega^1_{\mathcal{D}}$, the fluctuated Dirac operator $\mathcal{D}_A$ or $\mathcal{D}_{\widetilde{A}}$ is an operator with compact resolvent, and in particular its kernel is a finite dimensional space. This space is invariant by J and χ.*

Note that $\mathcal{U}(\mathcal{A})$ acts on $\mathcal{D}$ by $\mathcal{D} \to \mathcal{D}_u = u\mathcal{D}u^*$ leaving invariant the spectrum of $\mathcal{D}$. Since $\mathcal{D}_u = \mathcal{D} + u[\mathcal{D}, u^*]$ and in a C^*-algebra, any element a is a linear combination of at most four unitaries, Definition 6.8 is quite natural.

The inner automorphisms of a spectral triple correspond to inner fluctuation of the metric defined by (24).

One checks directly that a fluctuation of a fluctuation is a fluctuation and that the unitary group $\mathcal{U}(\mathcal{A})$ is gauge compatible for the adjoint representation but to be an inner fluctuation is not a symmetric relation. It can append that $\mathcal{D}_A = 0$ with $\mathcal{D} \neq 0$.

Lemma 6.10. *Let $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ be a spectral triple and $X \in \Psi(\mathcal{A})$. Then $⨍ X^* = \overline{⨍ X}$. If the spectral triple is real, then, for $X \in \Psi(\mathcal{A})$, $JXJ^{-1} \in \Psi(\mathcal{A})$ and $⨍ JXJ^{-1} = ⨍ X^* = \overline{⨍ X}$.*

If $A = A^*$, then for $k, l \in \mathbb{N}$, the integrals $⨍ A^l \mathcal{D}^{-k}$, $⨍ \left(A\mathcal{D}^{-1}\right)^k$, $⨍ A^l |\mathcal{D}|^{-k}$, $⨍ \chi A^l |\mathcal{D}|^{-k}$, $⨍ A^l \mathcal{D} |\mathcal{D}|^{-k}$ are real valued.

We remark that the fluctuations leave invariant the first term of the spectral action (33). This is a generalization of the fact that in the commutative case, the noncommutative integral depends only on the principal symbol of the Dirac operator $\mathcal{D}$ and this symbol is stable by adding a gauge potential like in $\mathcal{D} + A$. Note however that *the symmetrized gauge potential $A + \epsilon JAJ^{-1}$ is always zero in this commutative case for any selfadjoint one-form A.*

Theorem 6.11. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a regular spectral triple which is simple and of dimension d. Let $A \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ be a selfadjoint gauge potential. Then, $\zeta_{D_{\widetilde{A}}}(0) = \zeta_D(0) + \sum_{q=1}^{d} \frac{(-1)^q}{q} ⨍ (\widetilde{A}D^{-1})^q$.*

The proof of this result, necessary for spectral action computation, needs few preliminaries.

Definition 6.12. *For an operator T, define the one-parameter group and notation*

$$\sigma_z(T) := |D|^z T |D|^{-z},\ z \in \mathbb{C}. \qquad \epsilon(T) := \nabla(T) D^{-2},\ \textit{(recall that } \nabla(T) = [\mathcal{D}^2, T]).$$

The expansion of the one-parameter group σ_z gives for $T \in OP^q$

$$\sigma_z(T) \sim \sum_{r=0}^{N} g(z,r)\, \varepsilon^r(T) \mod OP^{-N-1+q} \tag{27}$$

where $g(z,r) := \frac{1}{r!}(\frac{z}{2}) \cdots (\frac{z}{2} - (r-1)) = \binom{z/2}{r}$ with the convention $g(z,0) := 1$.

We fix a regular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension d and a self-adjoint 1-form A. Despite previous remark before Lemma 6.10, we pay attention here to the kernel of $\mathcal{D}_A$ since this operator can be non-invertible even if $\mathcal{D}$ is, so we define

$$\mathcal{D}_A := \mathcal{D} + \widetilde{A} \text{ where } \widetilde{A} := A + \varepsilon JAJ^{-1}, \qquad D_A := \mathcal{D}_A + P_A \tag{28}$$

where P_A is the projection on $\operatorname{Ker} \mathcal{D}_A$. Remark that $\widetilde{A} \in \mathcal{D}(\mathcal{A}) \cap OP^0$ and $\mathcal{D}_A \in \mathcal{D}(\mathcal{A}) \cap OP^1$. We note $V_A := P_A - P_0$. and as the following lemma shows, V_A is a smoothing operator:

Lemma 6.13. *(i)* $\bigcap_{k \geq 1} \operatorname{Dom}(\mathcal{D}_A)^k \subseteq \bigcap_{k \geq 1} \operatorname{Dom} |D|^k$.

(ii) $\operatorname{Ker} \mathcal{D}_A \subseteq \bigcap_{k \geq 1} \operatorname{Dom} |D|^k$.

(iii) For any $\alpha, \beta \in \mathbb{R}$, $|D|^\beta P_A |D|^\alpha$ is bounded.

(iv) $P_A \in OP^{-\infty}$.

Let us define $X := \mathcal{D}_A^2 - \mathcal{D}^2 = \widetilde{A}\mathcal{D} + \mathcal{D}\widetilde{A} + \widetilde{A}^2$, $\quad X_V := X + V_A$, thus $X \in \mathcal{D}_1(\mathcal{A}) \cap OP^1$ and by Lemma 6.13, $X_V \sim X \mod OP^{-\infty}$. Now let $Y := \log(D_A^2) - \log(D^2)$ which makes sense since $D_A^2 = \mathcal{D}_A^2 + P_A$ is invertible for any A.
By definition of X_V, we get $Y = \log(D^2 + X_V) - \log(D^2)$.

Lemma 6.14. *Y is a pseudodifferential operator in OP^{-1} with the expansion for any $N \in \mathbb{N}$*

$$Y \sim \sum_{p=1}^{N} \sum_{k_1,\cdots,k_p=0}^{N-p} \tfrac{(-1)^{|k|_1+p+1}}{|k|_1+p} \nabla^{k_p}(X\nabla^{k_{p-1}}(\cdots X\nabla^{k_1}(X)\cdots))D^{-2(|k|_1+p)} \quad \bmod OP^{-N-1}.$$

For any $N \in \mathbb{N}$ and $s \in \mathbb{C}$,

$$|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^{N} K_p(Y,s)|D|^{-s} \quad \bmod OP^{-N-1-\Re(s)} \tag{29}$$

with $K_p(Y,s) \in OP^{-p}$. For any $p \in \mathbb{N}$ and $r_1,\cdots,r_p \in \mathbb{N}_0$, $\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y) \in \Psi(\mathcal{A})$.

Proof of Theorem 6.11. See [12]. Since the spectral triple is simple, equation (29) entails that $\zeta_{D_A}(0) - \zeta_D(0) = \mathrm{Tr}(K_1(Y,s)|D|^{-s})_{|s=0}$. Thus, with (27), we get $\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2}⨍ Y$. Now the conclusion follows from $⨍ \log\big((1+S)(1+T)\big) = ⨍ \log(1+S) + ⨍ \log(1+T)$ for $S,T \in \Psi(\mathcal{A}) \cap OP^{-1}$ (since $\log(1+S) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} S^n$) with $S = D^{-1}A$ and $T = AD^{-1}$; so $⨍ \log(1+XD^{-2}) = 2⨍ \log(1+AD^{-1})$ and $-\frac{1}{2}⨍ Y = \sum_{q=1}^{d} \frac{(-1)^q}{q} ⨍ (\widetilde{A}D^{-1})^q$. ∎

Lemma 6.15. *For any $k \in \mathbb{N}_0$,*

$$\operatorname*{Res}_{s=d-k} \zeta_{D_A}(s) = \operatorname*{Res}_{s=d-k} \zeta_D(s) + \sum_{p=1}^{k} \sum_{r_1,\cdots,r_p=0}^{k-p} \operatorname*{Res}_{s=d-k} h(s,r,p)\ \mathrm{Tr}\left(\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s}\right),$$

where $h(s,r,p) := (-s/2)^p \int_{0\le t_1\le\cdots\le t_p\le 1} g(-st_1,r_1)\cdots g(-st_p,r_p)\, dt$.

Our operators $|D_A|^k$ are pseudodifferential operators: for any $k \in \mathbb{Z}$, $|D_A|^k \in \Psi^k(\mathcal{A})$.

The following result is quite important since it shows that one can use $⨍$ for D or D_A:

Proposition 6.16. *If the spectral triple is simple, $\operatorname*{Res}_{s=0} \mathrm{Tr}\left(P|D_A|^{-s}\right) = ⨍ P$ for any pseudodifferential operator P. In particular, for any $k \in \mathbb{N}_0$ $⨍ |D_A|^{-(d-k)} = \operatorname*{Res}_{s=d-k} \zeta_{D_A}(s)$. Moreover,*

$$\begin{aligned} &⨍ |D_A|^{-d} = ⨍ |D|^{-d}, \\ &⨍ |D_A|^{-(d-1)} = ⨍ |D|^{-(d-1)} - (\tfrac{d-1}{2})⨍ X|D|^{-d-1}, \\ &⨍ |D_A|^{-(d-2)} = ⨍ |D|^{-(d-2)} + \tfrac{d-2}{2}\Big(-⨍ X|D|^{-d} + \tfrac{d}{4}⨍ X^2|D|^{-2-d}\Big). \end{aligned} \tag{30}$$

6.5 Tadpole

In [30], the following definition is introduced inspired by the quantum field theory.

Definition 6.17. *In $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the tadpole $Tad_{\mathcal{D}+A}(k)$ of order k, for $k \in \{\, d-l \,:\, l \in \mathbb{N}\,\}$ is the term linear in $A = A^* \in \Omega^1_{\mathcal{D}}$, in the Λ^k term of* (33) *where $\mathcal{D} \to \mathcal{D} + A$.*

If moreover, the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$ is real, the tadpole $Tad_{\mathcal{D}+\tilde{A}}(k)$ is the term linear in A, in the Λ^k term of (33) *where $\mathcal{D} \to \mathcal{D} + \widetilde{A}$.*

Proposition 6.18. *Let* $(\mathcal{A},\,\mathcal{H},\,\mathcal{D})$ *be a spectral triple of dimension d with simple dimension spectrum. Then*

$$\mathrm{Tad}_{\mathcal{D}+A}(d-k) = -(d-k)⨍ A\mathcal{D}|\mathcal{D}|^{-(d-k)-2}, \quad \forall k \neq d, \qquad \mathrm{Tad}_{\mathcal{D}+A}(0) = -⨍ A\mathcal{D}^{-1}.$$

Moreover, if the triple is real, $\mathrm{Tad}_{\mathcal{D}+\widetilde{A}} = 2\,\mathrm{Tad}_{\mathcal{D}+A}$.

Corollary 6.19. *In a real spectral triple* $(\mathcal{A},\mathcal{H},\mathcal{D})$*, if* $A = A^* \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ *is such that* $\widetilde{A} = 0$*, then* $\mathrm{Tad}_{D+A}(k) = 0$ *for any* $k \in \mathbb{Z}$*,* $k \leq d$*.*

The vanishing tadpole of order 0 has the following equivalence (see [12])

$$⨍ A\mathcal{D}^{-1} = 0,\, \forall A \in \Omega^1_{\mathcal{D}}(\mathcal{A}) \iff ⨍ ab = ⨍ a\alpha(b),\, \forall a,b \in \mathcal{A}, \qquad \alpha(b) := \mathcal{D}b\mathcal{D}^{-1}.$$

The existence of tadpoles is important since, for instance, $A = 0$ is not necessarily a stable solution of the classical field equation deduced from spectral action expansion, [50].

6.6 Commutative geometry

Definition 6.20. *Consider a commutative spectral triple given by a compact Riemannian spin manifold M of dimension d without boundary and its Dirac operator* $D\!\!\!/$ *associated to the Levi–Civita connection. This means* $\big(\mathcal{A} := C^\infty(M),\, \mathcal{H} := L^2(M,S),\, D\!\!\!/\big)$ *where S is the spinor bundle over M. This triple is real since, due to the existence of a spin structure, the charge conjugation operator generates an anti-linear isometry J on* $\mathcal{H}$ *such that* $JaJ^{-1} = a^*, \quad \forall a \in \mathcal{A}$*, and when d is even, the grading is given by the chirality matrix* $\chi := (-i)^{d/2}\,\gamma^1\gamma^2\cdots\gamma^d$*. Such triple is said to be a commutative geometry.*

In the polynomial algebra $\mathcal{D}(\mathcal{A})$ of Definition 6.6, we added $\mathcal{A}^\circ$. In the commutative case, $\mathcal{A}^\circ \simeq J\mathcal{A}J^{-1} \simeq \mathcal{A}$ which also gives $JAJ^{-1} = -\epsilon\, A^*, \quad \forall A \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ or $\widetilde{A} = 0$ when $A = A^*$.

As noticed by Wodzicki, $⨍ P$ is equal to -2 times the coefficient in $\log t$ of the asymptotics of $\mathrm{Tr}(P\,e^{-t\,D\!\!\!/^2})$ as $t \to 0$. It is remarkable that this coefficient is independent of $D\!\!\!/$ as seen in Theorem 2.18 and this gives a close relation between the ζ function and heat kernel expansion with *WRes*. Actually, by [47, Theorem 2.7]

$$\mathrm{Tr}(P\,e^{-t\,D\!\!\!/^2}) \underset{t\downarrow 0^+}{\sim} \textstyle\sum_{k=0}^\infty a_k\, t^{(k-ord(P)-d)/2} + \sum_{k=0}^\infty(-a'_k\,\log t + b_k)\,t^k,$$

so $⨍ P = 2a'_0$. Since $⨍$, *WRes* are traces on $\Psi\big(C^\infty(M)\big)$, Corollary 2.19 gives $⨍ P = c\;WRes\;P$. Because $\mathrm{Tr}(P\,D\!\!\!/^{-2s}) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\;\mathrm{Tr}(P\,e^{-t\,D\!\!\!/^2})\,dt$, the non-zero coefficient a'_k, $k \neq 0$ creates a pole of $\mathrm{Tr}(P\,D\!\!\!/^{-2s})$ of order $k+2$ as we get $\int_0^1 t^{s-1}\log(t)^k\,dt = \frac{(-1)^k k!}{s^{k+1}}$ and $\Gamma(s) = \frac{1}{s}+\gamma+s\,g(s)$ where γ is the Euler constant and the function g is also holomorphic around zero.

Proposition 6.21. *Let* $Sp(M)$ *be the dimension spectrum of a commutative geometry of dimension d. Then* $Sp(M)$ *is simple and* $Sp(M) = \{\,d-k \mid k \in \mathbb{N}\,\}$*.*

Remark 6.22. *When the dimension spectrum is not simple, the analog of WRes is no longer a trace.*

The equation (30) *can be obtained via* (8) *as* $\sigma_d^{|\mathcal{D}_A|^{-d}} = \sigma_d^{|\mathcal{D}|^{-d}}$.

In dimension $d = 4$*, the computation in* (19) *of coefficient* $a_4(1,\mathcal{D}_A^2)$ *gives*

$$\zeta_{\mathcal{D}_A}(0) = c_1\int_M (5R^2 - 8R\mu\nu r^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,dvol + c_2\int_M tr(F_{\mu\nu}F^{\mu\nu})\,dvol,$$

see Corollary 7.4 to see precise correspondence between $a_k(1,\mathcal{D}_A^2)$ *and* $\zeta_{\mathcal{D}_A}(0)$*. One recognizes the Yang–Mills action which will be generalized in Section 7.1.3 to arbitrary spectral triples.*

According to Corollary 6.19, a commutative geometry has no tadpoles [58].

6.7 Scalar curvature

What could be the scalar curvature of a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$? Of course, we consider first the case of a commutative geometry of dimension $d=4$: if s is the scalar curvature and $f\in C^\infty(M)$, we know that $⨍ f(x)\not{D}^{-d+2}=\int_M f(x)\,s(x)\,dvol(x)$. This suggests the

Definition 6.23. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a spectral triple of dimension d. The scalar curvature is the map $\mathcal{R}: a\in\mathcal{A}\to\mathbb{C}$ defined by $\mathcal{R}(a):=⨍ a\,\mathcal{D}^{-d+2}$.*

In the commutative case, $\mathcal{R}$ is a trace on the algebra. More generally

Proposition 6.24. *[30, Proposition 1.153] If $\mathcal{R}$ is a trace on $\mathcal{A}$ and the tadpoles $⨍ A\,\mathcal{D}^{-d+1}$ are zero for all $A\in\Omega^1_{\mathcal{D}}$, $\mathcal{R}$ is invariant by inner fluctuations $\mathcal{D}\to\mathcal{D}+A$.*

6.8 Tensor product of spectral triples

There is a natural notion of tensor for spectral triples which corresponds to direct product of manifolds in the commutative case. Let $(\mathcal{A}_i,\mathcal{D}_i,\mathcal{H}_i)$, $i=1,2$, be two spectral triples of dimension d_i with simple dimension spectrum. Assume the first to be of even dimension, with grading χ_1. The spectral triple $(\mathcal{A},\mathcal{D},\mathcal{H})$ associated to the tensor product is defined by

$$\mathcal{A}:=\mathcal{A}_1\otimes\mathcal{A}_2,\quad \mathcal{D}:=\mathcal{D}_1\otimes 1+\chi_1\otimes\mathcal{D}_2,\quad \mathcal{H}:=\mathcal{H}_1\otimes\mathcal{H}_2.$$

The interest of χ_1 is to guarantee additivity: $\mathcal{D}^2=\mathcal{D}_1^2\otimes 1+1\otimes\mathcal{D}_2^2$.

Lemma 6.25. *Assume that $\mathrm{Tr}(e^{-t\mathcal{D}_1^2})\sim_{t\to 0} a_1\,t^{-d_1/2}$ and $\mathrm{Tr}(e^{-t\mathcal{D}_2^2})\sim_{t\to 0} a_2\,t^{-d_2/2}$.*
The triple $(\mathcal{A},\mathcal{D},\mathcal{H})$ has dimension $d=d_1+d_2$.
Moreover, the function $\zeta_{\mathcal{D}}(s)=\mathrm{Tr}(|D|^{-s})$ has a simple pole at $s=d_1+d_2$ with

$$\mathrm{Res}_{s=d_1+d_2}\Big(\zeta_{\mathcal{D}}(s)\Big)=\tfrac{1}{2}\,\tfrac{\Gamma(d_1/2)\Gamma(d_2/2)}{\Gamma(d/2)}\,\mathrm{Res}_{s=d_1}\Big(\zeta_{\mathcal{D}_1}(s)\Big)\;\mathrm{Res}_{s=d_2}\Big(\zeta_{\mathcal{D}_2}(s)\Big).$$

Proof. We get $\zeta_{\mathcal{D}}(2s)=\sum_{n=0}^\infty \mu_n(D_1^2\otimes 1+1\otimes\mathcal{D}_2^2)^{-s}=\sum_{n,m=0}^\infty\Big(\mu_n(\mathcal{D}_1^2)+\mu_m(\mathcal{D}_2^2)\Big)^{-s}$. Since $\Big(\mu_n(\mathcal{D}_1)^2+\mu_m(\mathcal{D}_2)^2\Big)^{-(c_1+c_2)/2}\leq\mu_n(\mathcal{D}_1)^{-c_1}\mu_m(\mathcal{D}_2)^{-c_2}$, $\zeta_{\mathcal{D}}(c_1+c_2)\leq\zeta_{\mathcal{D}_1}(c_1)\zeta_{\mathcal{D}_2}(c_2)$ if $c_i>d_i$, so $d:=\inf\{\,c\in\mathbb{R}^+\,:\,\zeta_{\mathcal{D}}(c)<\infty\,\}\leq d_1+d_2$. We claim that $d=d_1+d_2$: recall first that $a_i:=\underset{s=d_i/2}{\mathrm{Res}}\Big(\Gamma(s)\zeta_{\mathcal{D}_i}(2s)\Big)=\Gamma(d_i/2)\underset{s=d_i/2}{\mathrm{Res}}\Big(\zeta_{\mathcal{D}_i}(2s)\Big)=\tfrac{1}{2}\Gamma(d_i/2)\underset{s=d_i}{\mathrm{Res}}\Big(\zeta_{\mathcal{D}_i}(s)\Big)$. If $f(s):=\Gamma(s)\,\zeta_D(2s)$, $f(s)=\int_0^1\mathrm{Tr}\left(e^{-t\mathcal{D}^2}\right)t^{s-1}\,dt+g(s)=\int_0^1\mathrm{Tr}\left(e^{-t\mathcal{D}_1^2}\right)\mathrm{Tr}\left(e^{-t\mathcal{D}_2^2}\right)t^{s-1}\,dt+g(s)$ where g is holomorphic since the map $x\in\mathbb{R}\to\int_1^\infty e^{-tx^2}t^{x-1}\,dt$ is in Schwartz space.

Since $\mathrm{Tr}\left(e^{-t\mathcal{D}_1^2}\right)\mathrm{Tr}\left(e^{-t\mathcal{D}_2^2}\right)\sim_{t\to 0} a_1a_2\,t^{-(d_1+d_2)/2}$, the function $f(s)$ has a simple pole at $s=(d_1+d_2)/2$. We conclude that $\zeta_{\mathcal{D}}(s)$ has a simple pole at $s=d_1+d_2$. Moreover, using a_i, $\tfrac{1}{2}\Gamma((d_1+d_2)/2)\underset{s=d}{\mathrm{Res}}\Big(\zeta_{\mathcal{D}}(s)\Big)=\tfrac{1}{2}\Gamma(d_1/2)\underset{s=d_1}{\mathrm{Res}}\Big(\zeta_{\mathcal{D}_1}(s)\Big)\,\tfrac{1}{2}\Gamma(d_2/2)\underset{s=d_2}{\mathrm{Res}}\Big(\zeta_{\mathcal{D}_2}(s)\Big)$. □

7 Spectral action

7.1 On the search for a good action functional

We would like to obtain a good action for any spectral triple and for this it is useful to look at some examples in physics. In any physical theory based on geometry, the interest of an action functional is, by a minimization process, to exhibit a particular geometry, for instance, trying to distinguish between different metrics. This is the case in general relativity with the Einstein–Hilbert action (with its Riemannian signature).

7.1.1 Einstein–Hilbert action

This action is $S_{EH}(g) := -\int_M s_g(x)\, dvol_g(x)$ where s is the scalar curvature (chosen positive for the sphere). Up to a constant, this is $⨍ \not{D}^{-2}$ in dimension 4 as quoted after (21).
This action is interesting for the following reason: Let $\mathcal{M}_1$ be the set of Riemannian metrics g on M such that $\int_M dvol_g = 1$. By a theorem of Hilbert [5], $g \in \mathcal{M}_1$ is a critical point of $S_{EH}(g)$ restricted to $\mathcal{M}_1$ if and only if (M, g) is an Einstein manifold (the Ricci curvature R of g is proportional by a constant to g: $R = c\,g$). Taking the trace, this means that $s_g = c \dim(M)$ and such manifold have a constant scalar curvature.

But in the search for invariants under diffeomorphisms, they are more quantities than the Einstein–Hilbert action, a trivial example being $\int_M f\big(s_g(x)\big)\, dvol_g(x)$ and they are others [42]. In this desire to implement gravity in noncommutative geometry, the eigenvalues of the Dirac operator look as natural variables [69]. However we are looking for observables which add up under disjoint unions of different geometries.

7.1.2 Quantum approach and spectral action

In a way, a spectral triple fits quantum field theory since $\mathcal{D}^{-1}$ can be seen as the propagator (or line element ds) for (Euclidean) fermions and we can compute Feynman graphs with fermionic internal lines. As glimpsed in section 6.4, the gauge bosons are only derived objects obtained from internal fluctuations via Morita equivalence given by a choice of a connection which is associated to a one-form in $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Thus, the guiding principle followed by Connes and Chamseddine is to use a theory which is pure gravity with a functional action based on the spectral triple, namely which depends on the spectrum of $\mathcal{D}$ [10]. They proposed the

Definition 7.1. *The spectral action of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is defined by $\mathcal{S}(\mathcal{D}, f, \Lambda) := \operatorname{Tr}\big(f(\mathcal{D}^2/\Lambda^2)\big)$ where $\Lambda \in \mathbb{R}^+$ plays the role of a cut-off and f is any positive function (such that $f(\mathcal{D}^2/\Lambda^2)$ is a trace-class operator).*

Remark 7.2. *We can also define $\mathcal{S}(\mathcal{D}, f, \Lambda) = \operatorname{Tr}\big(f(\mathcal{D}/\Lambda)\big)$ when f is positive and even. With this second definition, $S(\mathcal{D}, g, \Lambda) = \operatorname{Tr}\big(f(\mathcal{D}^2/\Lambda^2)\big)$ with $g(x) := f(x^2)$.*

For f, one can think of the characteristic function of $[-1, 1]$, thus $f(\mathcal{D}/\Lambda)$ is nothing else but the number of eigenvalues of $\mathcal{D}$ within $[-\Lambda, \Lambda]$.

When this action has an asymptotic series in $\Lambda \to \infty$, we deal with an effective theory. Naturally, $\mathcal{D}$ has to be replaced by $\mathcal{D}_A$ which is a just a decoration. To this bosonic part of the action, one adds a fermionic term $\frac{1}{2}\langle J\psi, \mathcal{D}\psi\rangle$ for $\psi \in \mathcal{H}$ to get a full action. In the standard model of particle physics, this latter corresponds to the integration of the Lagrangian part for the coupling between gauge bosons and Higgs bosons with fermions. Actually, the finite dimension part of the noncommutative standard model is of KO-dimension 6, thus $\langle \psi, \mathcal{D}\psi\rangle$ has to be replaced by $\frac{1}{2}\langle J\psi, \mathcal{D}\psi\rangle$ for $\psi = \chi\psi \in \mathcal{H}$, see [30].

7.1.3 Yang–Mills action

This action plays an important role in physics so recall first the classical situation: let G be a compact Lie group with its Lie algebra $\mathfrak{g}$ and let $A \in \Omega^1(M, \mathfrak{g})$ be a connection. If $F := da + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$ is the curvature (or field strength) of A, then the Yang-Mills action is $S_{YM}(A) = \int_M \operatorname{tr}(F\wedge\star F)\, dvol_g$. In the abelian case $G = U(1)$, it is the Maxwell action and its quantum version is the quantum electrodynamics (QED) since the un-gauged $U(1)$ of electric charge conservation can be gauged and its gauging produces electromagnetism [97]. It is conformally invariant when $\dim(M) = 4$.

The study of its minima and its critical values can also been made for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension d [23, 24]: let $A \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ and curvature $\theta = dA + A^2$; then it is natural to consider $I(A) := \operatorname{Tr}_{Dix}(\theta^2|\mathcal{D}|^{-d})$ since it coincides (up to a constant) with the previous Yang-Mills action in the commutative case: if $P = \theta^2|\mathcal{D}|^{-d}$, then Theorems 2.18 and 3.11 give the claim since for the principal symbol, $\operatorname{tr}\big(\sigma^P(x,\xi)\big) = c\operatorname{tr}(F \wedge \star F)(x)$.

There is nevertheless a problem with the definition of dA: if $A = \sum_j \pi(a_j)[\mathcal{D}, \pi(b_j)]$, then $dA = \sum_j [\mathcal{D}, \pi(a_j)][\mathcal{D}, \pi(b_j)]$ can be non-zero while $A = 0$. This ambiguity means that, to get a graded differential algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$, one must divide by a junk, for instance $\Omega^2_{\mathcal{D}} \simeq \pi(\Omega^2/\pi\big(\delta(\operatorname{Ker}(\pi) \cap \Omega^1)\big)$ where $\Omega^k(\mathcal{A})$ is the set of universal k-forms over $\mathcal{A}$ given by the set of $a_0\delta a_1 \cdots \delta a_k$ (before representation on $\mathcal{H}$: $\pi(a_0\delta a_1 \cdots \delta a_k) := a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k]$). Let $\mathcal{H}_k$ be the Hilbert space completion of $\pi(\Omega^k(\mathcal{A}))$ with the scalar product defined by $\langle A_1, A_2\rangle_k := \operatorname{Tr}_{Dix}(A_2^* A_1|\mathcal{D}|^{-d})$ for $A_j \in \pi(\Omega^k(\mathcal{A}))$.
The Yang–Mills action on $\Omega^1(\mathcal{A})$ is $S_{YM}(V) := \langle \delta V + V^2, \delta V + V^2\rangle$. It is positive, quartic and gauge invariant under $V \to \pi(u)V\pi(u^*) + \pi(u)[\mathcal{D}, \pi(u^*)]$ when $u \in \mathcal{U}(\mathcal{A})$. Moreover, $S_{YM}(V) = \inf\{\, I(\omega) \,|\, \omega \in \Omega^1(\mathcal{A}),\, \pi(\omega) = V\,\}$ since the above ambiguity disappears when taking the infimum.

This Yang–Mills action can be extended to the equivalent of Hermitean vector bundles on M, namely finitely projective modules over $\mathcal{A}$.

The spectral action is more conceptual than the Yang–Mills action since it gives no fundamental role to the distinction between gravity and matter in the artificial decomposition $\mathcal{D}_A = \mathcal{D} + A$. For instance, for the minimally coupled standard model, the Yang–Mills action for the vector potential is part of the spectral action, as far as the Einstein–Hilbert action for the Riemannian metric [11].

As quoted in [16], the spectral action has conceptual advantages:

- Simplicity: when f is a cutoff function, the spectral action is just the counting function.

- Positivity: when f is positive (which is the case for a cutoff function), the action $\operatorname{Tr}\big(f(\mathcal{D}/\Lambda)\big) \geq 0$ has the correct sign for a Euclidean action: the positivity of the function f will insure that the actions for gravity, Yang-Mills, Higgs couplings are all positive and the Higgs mass term is negative.

- Invariance: the spectral action has a much stronger invariance group than the usual diffeomorphism group as for the gravitational action; this is the unitary group of the Hilbert space $\mathcal{H}$.

However, this action is not local but becomes local when replaced by its asymptotic expansion:

7.2 Asymptotic expansion for $\Lambda \to \infty$

The heat kernel method already used in previous sections will give a control of spectral action $S(\mathcal{D}, f, \Lambda)$ when Λ goes to infinity.

Theorem 7.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with a simple dimension spectrum Sd.*

We assume that

$$\operatorname{Tr}\left(e^{-t\mathcal{D}^2}\right) \underset{t\downarrow 0}{\sim} \sum_{\alpha\in Sd} a_\alpha\, t^\alpha \quad \textit{with } a_\alpha \neq 0. \tag{31}$$

Then, for the zeta function $\zeta_\mathcal{D}$ defined in (25)

$$a_\alpha = \tfrac{1}{2}\operatorname{Res}_{s=-2\alpha}\Big(\Gamma(s/2)\zeta_\mathcal{D}(s)\Big). \tag{32}$$

(i) If $\alpha < 0$, $\zeta_\mathcal{D}$ has a pole at -2α with $a_\alpha = \frac{1}{2}\Gamma(-\alpha)\underset{s=-2\alpha}{\operatorname{Res}}\,\zeta_\mathcal{D}(s)$.

(ii) For $\alpha = 0$, we get $a_0 = \zeta_\mathcal{D}(0) + \dim \operatorname{Ker} \mathcal{D}$.

(iii) If $\alpha > 0$, $a_\alpha = \zeta(-2\alpha)\underset{s=-\alpha}{\operatorname{Res}}\,\Gamma(s)$.

(iv) The spectral action has the asymptotic expansion over the positive part Sd^+ of Sd:

$$\operatorname{Tr}\Big(f(\mathcal{D}/\Lambda)\Big) \underset{\Lambda\to+\infty}{\sim} \sum_{\beta\in Sd^+} f_\beta\, \Lambda^\beta \int\!\!\!\!\!\!- |\mathcal{D}|^\beta + f(0)\,\zeta_\mathcal{D}(0) + \cdots \tag{33}$$

where the dependence of the even function f is $f_\beta := \int_0^\infty f(x)\,x^{\beta-1}\,dx$ and $\cdots$ involves the full Taylor expansion of f at 0.

Proof. (i) Since $\Gamma(s/2)\,|\mathcal{D}|^{-s} = \int_0^\infty e^{-t\mathcal{D}^2}\,t^{s/2-1}\,dt = \int_0^1 e^{-t\mathcal{D}^2}\,t^{s/2-1}\,dt + f(s)$, where the function f is holomorphic (since the map $x \to \int_1^\infty e^{-tx^2}x^{s/2-1}\,dt$ is in the Schwartz space), the swap of $\operatorname{Tr}\left(e^{-t\mathcal{D}^2}\right)$ with a sum of $a_\alpha\, t^\alpha$ and $a_\alpha \int_0^1 t^{\alpha+s/2-1}\,dt = \frac{2\alpha_a}{s+2\alpha}$ yields (32).

(ii) The regularity of $\Gamma(s/2)^{-1} \underset{s\to 0}{\sim} s/2$ around zero implies that only the pole part at $s=0$ of $\int_0^\infty \operatorname{Tr}\left(e^{-t\mathcal{D}^2}\right)t^{s/2-1}\,dt$ contributes to $\zeta_\mathcal{D}(0)$. This contribution is $a_0\int_0^1 t^{s/2-1}\,dt = \frac{2a_0}{s}$.

(iii) follows from (32).

(iv) Assume $f(x) = g(x^2)$ where g is a Laplace transform: $g(x) := \int_0^\infty e^{-sx}\,\phi(s)\,ds$. We will see in Section 7.3 how to relax this hypothesis.
Since $g(t\mathcal{D}^2) = \int_0^\infty e^{-st\mathcal{D}^2}\,\phi(s)\,ds$, $\operatorname{Tr}\left(g(t\mathcal{D}^2)\right) \underset{t\downarrow 0}{\sim} \sum_{\alpha\in \mathrm{Sp}^+} a_\alpha\, t^\alpha \int_0^\infty s^\alpha\, g(s)\,ds$. When $\alpha < 0$, $s^\alpha = \Gamma(-\alpha)^{-1}\int_0^\infty e^{-sy}\,y^{-\alpha-1}\,dy$ and $\int_0^\infty s^\alpha\,\phi(s)\,ds = \Gamma(-\alpha)^{-1}\int_0^\infty g(y)\,y^{-\alpha-1}\,dy$. Thus

$$\operatorname{Tr}\left(g(t\mathcal{D}^2)\right) \underset{t\downarrow 0}{\sim} \sum_{\alpha\in \mathrm{Sp}^-}\Big[\tfrac{1}{2}\underset{s=-2\alpha}{\operatorname{Res}}\,\zeta_\mathcal{D}(s)\int_0^\infty g(y)\,y^{-\alpha-1}\,dy\Big]\,t^\alpha.$$

Thus (33) follows from (i), (ii) and $\frac{1}{2}\int_0^\infty g(y)\,y^{\beta/2-1}\,dy = \int_0^\infty f(x)\,x^{\beta-1}\,dx$. □

It can be useful to make a connection with (20) of Section 5.2:

Corollary 7.4. *Assume that the spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ has dimension d. If*

$$\operatorname{Tr}\Big(e^{-t\,\mathcal{D}^2}\Big) \underset{t\downarrow 0}{\sim} \sum_{k\in\{0,\cdots,d\}} t^{(k-d)/2}\,a_k(\mathcal{D}^2) + \cdots, \tag{34}$$

then $\mathcal{S}(\mathcal{D},f,\Lambda) \underset{t\downarrow 0}{\sim} \sum_{k\in\{1,\cdots,d\}} f_k\,\Lambda^k\,a_{d-k}(\mathcal{D}^2) + f(0)\,a_d(\mathcal{D}^2) + \cdots$ *with* $f_k := \frac{1}{\Gamma(k/2)}\int_0^\infty f(s)s^{k/2-1}ds$. *Moreover,*

$$a_k(\mathcal{D}^2) = \tfrac{1}{2}\,\Gamma(\tfrac{d-k}{2})\int\!\!\!\!\!\!- |\mathcal{D}|^{-d+k} \textit{ for } k = 0,\cdots,d-1, \qquad a_d(\mathcal{D}^2) = \dim \operatorname{Ker}\mathcal{D} + \zeta_{\mathcal{D}^2}(0). \tag{35}$$

The asymptotics (33) uses the value of $\zeta_\mathcal{D}(0)$ in the constant term Λ^0, so it is fundamental to look at its variation under a gauge fluctuation $\mathcal{D} \to \mathcal{D} + A$ as we saw in Theorem 6.11.

7.3 Remark on the use of Laplace transform

The spectral action asymptotic behavior $S(\mathcal{D}, f, \Lambda) \underset{\Lambda\to+\infty}{\sim} \sum_{n=0}^{\infty} c_n\, \Lambda^{d-n}\, a_n(\mathcal{D}^2)$ has been proved for a smooth function f which is a Laplace transform for an arbitrary spectral triple (with simple dimension spectrum) satisfying (31). However, this hypothesis is too restrictive since it does not cover the heat kernel case where $f(x) = e^{-x}$.

When the triple is commutative and $\mathcal{D}^2$ is a generalized Laplacian on sections of a vector bundle over a manifold of dimension 4, Estrada–Gracia-Bondía–Várilly proved in [37] that previous asymptotics is

$$\operatorname{Tr}\Big(f(\mathcal{D}^2/\Lambda^2)\Big) \sim \tfrac{1}{(4\pi)^2}\Big[\mathrm{rk(E)} \int_0^\infty x f(x)\, dx\, \Lambda^4 + b_2(\mathcal{D}^2) \int_0^\infty f(x)\, dx\, \Lambda^2 \\ + \sum_{m=0}^{\infty} (-1)^m\, f^{(m)}(0)\, b_{2m+4}(\mathcal{D}^2)\, \Lambda^{-2m}\Big], \quad \Lambda \to \infty$$

where $(-1)^m b_{2m+4}(\mathcal{D}^2) = \frac{(4\pi)^2}{m!}\, \mu_m(\mathcal{D}^2)$ are suitably normalized, integrated moment terms of the spectral density of $\mathcal{D}^2$.

The main point is that *this asymptotics makes sense in the Cesàro sense* (see [37] for definition) for f in $\mathcal{K}'(\mathbb{R})$, which is the dual of $\mathcal{K}(\mathbb{R})$. This latter is the space of smooth functions ϕ such that for some $a \in \mathbb{R}$, $\phi^{(k)}(x) = \mathcal{O}(|x|^{a-k})$ as $|x| \to \infty$, for each $k \in \mathbb{N}$. In particular, the Schwartz functions are in $\mathcal{K}(\mathbb{R})$ (and even dense).

Of course, the counting function is not smooth but is in $\mathcal{K}'(\mathbb{R})$, so the given asymptotic behavior is wrong beyond the first term, but is correct in the Cesàro sense. Actually there are more derivatives of f at 0 as explained on examples in [37, p. 243].

7.4 About convergence and divergence, local and global aspects of the asymptotic expansion

The asymptotic expansion series (34) of the spectral action may or may not converge. It is known that each function $g(\Lambda^{-1})$ defines at most a unique expansion series when $\Lambda \to \infty$ but the converse is not true since several functions have the same asymptotic series. We give here examples of convergent and divergent series of this kind.

When $M = \mathbb{T}^d$ as in Example 3.10 with $\Delta = \delta^{\mu\nu}\partial_\mu\partial_\nu$, $\operatorname{Tr}(e^{t\Delta}) = \frac{(4\pi)^{-d/2}\,\mathrm{Vol}(\mathbb{T}^d)}{t^{d/2}} + \mathcal{O}(t^{-d/2}\, e^{-1/4t})$, so the asymptotic series $\operatorname{Tr}(e^{t\Delta}) \simeq \frac{(4\pi)^{-d/2}\,\mathrm{Vol}(T^d)}{t^{d/2}}$, $t \to 0$, has only one term.

In the opposite direction, let now M be the unit four-sphere $\mathbb{S}^4$ and $\not{D}$ be the usual Dirac operator. By Proposition 6.21, equation (31) yields (see [19]):

$$\operatorname{Tr}(e^{-t\not{D}^2}) = \tfrac{1}{t^2}\Big(\tfrac{2}{3} + \tfrac{2}{3}\,t + \sum_{k=0}^{n} a_k\, t^{k+2} + \mathcal{O}(t^{n+3})\Big), \qquad a_k := \tfrac{(-1)^k\, 4}{3\,k!}\Big(\tfrac{B_{2k+2}}{2k+2} - \tfrac{B_{2k+4}}{2k+4}\Big)$$

with Bernoulli numbers B_{2k}. Thus $t^2 \operatorname{Tr}(e^{-t\not{D}^2}) \simeq \frac{2}{3} + \frac{2}{3}\,t + \sum_{k=0}^{\infty} a_k\, t^{k+2}$ when $t \to 0$ and this series is a not convergent but only asymptotic:

$a_k > \frac{4}{3\,k!}\frac{|B_{2k+4}|}{2k+4} > 0$ and $|B_{2k+4}| = 2\,\frac{(2k+4)!}{(2\pi)^{2k+4}}\,\zeta(2k+4) \simeq 4\sqrt{\pi(k+2)}\,\big(\frac{k+2}{\pi e}\big)^{2k+4} \to \infty$ if $k \to \infty$.

More generally, in the commutative case considered above and when $\mathcal{D}$ is a differential operator—like a Dirac operator, the coefficients of the asymptotic series of $\operatorname{Tr}(e^{-t\mathcal{D}^2})$ are locally defined by the symbol of $\mathcal{D}^2$ at point $x \in M$ but this is not true in general: in [45] is given a positive elliptic pseudodifferential such that non-locally computable coefficients especially appear in (34) when $2k > d$. Nevertheless, all coefficients are local for $2k \leq d$.

Recall that a locally computable quantity is the integral on the manifold of a local frame-independent smooth function of one variable, depending only on a finite number of derivatives of a finite number of terms in the asymptotic expansion of the total symbol of $\mathcal{D}^2$. For instance, some nonlocal information contained in the ultraviolet asymptotics can be recovered if one looks at the (integral) kernel of $e^{-t\sqrt{-\Delta}}$: in $\mathbb{T}^1$, with $\mathrm{Vol}(\mathbb{T}^1) = 2\pi$, we get [38]

$$\mathrm{Tr}(e^{-t\sqrt{-\Delta}}) = \tfrac{\sinh(t)}{\cosh(t)-1} = \coth(\tfrac{t}{2}) = \tfrac{2}{t}\sum_{k=0}^{\infty} \tfrac{B_{2k}}{(2k)!}\, t^{2k} = \tfrac{2}{t}[1 + \tfrac{t^2}{12} - \tfrac{t^4}{720} + \mathcal{O}(t^6)]$$

so the series converges when $t < 2\pi$ as $\frac{B_{2k}}{(2k)!} = (-1)^{k+1}\,\frac{2\,\zeta(2k)}{(2\pi)^{2k}}$, thus $\frac{|B_{2k}|}{(2k)!} \simeq \frac{2}{(2\pi)^{2k}}$ when $k \to \infty$.

Thus we have an example where $t \to \infty$ cannot be used with the asymptotic series.

Thus the spectral action of Corollary 7.4 precisely encodes these local and nonlocal behavior which appear or not in its asymptotics for different f. The coefficient of the action for the positive part (at least) of the dimension spectrum correspond to renormalized traces, namely the noncommutative integrals of (35). In conclusion, the asymptotics of spectral action may or may not have nonlocal coefficients.

For the flat torus $\mathbb{T}^d$, the difference between $\mathrm{Tr}(e^{t\Delta})$ and its asymptotic series is an term which is related to periodic orbits of the geodesic flow on $\mathbb{T}^d$. Similarly, the counting function $N(\lambda)$ (number of eigenvalues including multiplicities of Δ less than λ) obeys Weyl's law: $N(\lambda) = \frac{(4\pi)^{-d/2}\,\mathrm{Vol}(\mathbb{T}^d)}{\Gamma(d/2+1)}\,\lambda^{d/2} + o(\lambda^{d/2})$ — see [1] for a nice historical review on these fundamental points. The relationship between the asymptotic expansion of the heat kernel and the formal expansion of the spectral measure is clear: the small-t asymptotics of heat kernel is determined by the large-λ asymptotics of the density of eigenvalues (and eigenvectors). However, the latter is defined modulo some average: Cesàro sense as reminded in Section 7.3, or Riesz mean of the measure which washes out ultraviolet oscillations, but also gives informations on intermediate values of λ [38].

In [16, 76] are examples of spectral actions on commutative geometries of dimension 4 whose asymptotics have only two terms. In the quantum group $SU_q(2)$, the spectral action itself has only 4 terms, independently of the choice of function f. See [62] for more examples.

7.5 On the physical meaning of the asymptotics of spectral action

The spectral action is non-local. Its localization does not cover all situations: consider for instance the commutative geometry of a spin manifold M of dimension 4. One adds a gauge connection $A \in \Gamma^\infty\big(M, End(S)\big)$ to $\not{D}$ such that $\mathcal{D} = i\gamma^\mu(\partial_\mu + A_\mu)$, thus with a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. We apply (18) with $P = \mathcal{D}^2$ and find the coefficients $a_i(1, P)$ of (19) with $i = 0, 2, 4$. The asymptotic expansion corresponds to a weak field expansion.

Moreover a commutative geometry times a finite one where the finite one is algebra is a sum of matrices has been deeply and intensively investigated for the noncommutative approach to standard model of particle physics, see [20, 30]. This approach offers a lot of interesting perspectives, for instance, the possibility to compute the Higgs representations and mass (for each noncommutative model) is particularly instructive [10, 15, 17, 57, 63, 64, 71, 79]. Of course, since the first term in (33) is a cosmological term, one may be worried by its large value (for instance in the noncommutative standard model where the cutoff is, roughly speaking the Planck scale). At the classical level, one can work with unimodular gravity where the metric (so the Dirac operator) $\mathcal{D}$ varies within the set $\mathcal{M}_1$ of metrics which preserve the volume as in Section 7.1.1. Thus it remains only (!) to control the inflaton: see [13].

The spectral action has been computed in [60] for the quantum group $SU_q(2)$ which is not a deformation of $SU(2)$ of the type considered on the Moyal plane. It is quite peculiar since (33) has only a finite number of terms.

Due to the difficulties to deal with non-compact manifolds, the case of spheres $\mathbb{S}^4$ or $\mathbb{S}^3 \times \mathbb{S}^1$ has been investigated in [16,19] for instance in the case of Robertson–Walker metrics.

All the machinery of spectral geometry as been recently applied to cosmology, computing the spectral action in few cosmological models related to inflation, see [66, 76–78, 81, 95].

Spectral triples associated to manifolds with boundary have been considered in [14,18,18, 58,59,61]. The main difficulty is precisely to put nice boundary conditions to the operator $\mathcal{D}$ to still get a selfadjoint operator and then, to define a compatible algebra $\mathcal{A}$. This is probably a must to obtain a result in a noncommutative Hamiltonian theory in dimension 1+3.

The case of manifolds with torsion has also been studied in [53, 84, 85], and even with boundary in [61]. These works show that the Holst action appears in spectral actions and that torsion could be detected in a noncommutative world.

8 The noncommutative torus

The aim of this section is to compute the spectral action of the noncommutative torus. Due to a fundamental appearance of small divisors, the number theory is involved via a Diophantine condition. As a consequence, the result which essentially says that the spectral action of the noncommutative torus coincide with the action of the ordinary torus (up few constants) is awfully technical and this shows how life can be hard in noncommutative geometry!

Reference: [36].

8.1 Definition of the nc-torus

Let $C^\infty(\mathbb{T}^n_\Theta)$ be the smooth noncommutative n-torus associated to a non-zero skew-symmetric deformation matrix $\Theta \in M_n(\mathbb{R})$. It was introduced by Rieffel [93] and Connes [21] to generalize the n-torus $\mathbb{T}^n$. This means that $C^\infty(\mathbb{T}^n_\Theta)$ is the algebra generated by n unitaries u_i, $i = 1, \dots, n$ subject to the relations $u_l\, u_j = e^{i\,\Theta_{lj}}\, u_j\, u_l$, and with Schwartz coefficients: an element $a \in C^\infty(\mathbb{T}^n_\Theta)$ can be written as $a = \sum_{k\in\mathbb{Z}^n} a_k\, U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$ with the Weyl elements defined by $U_k := e^{-\frac{i}{2}k.\chi k}\, u_1^{k_1} \cdots u_n^{k_n}$, $k \in \mathbb{Z}^n$, the constraint relation reads

$$U_k U_q = e^{-\frac{i}{2}k.\Theta q}\, U_{k+q}, \text{ and } U_k U_q = e^{-ik.\Theta q}\, U_q U_k$$

where χ is the matrix restriction of Θ to its upper triangular part. Thus unitary operators U_k satisfy $U_k^* = U_{-k}$ and $[U_k, U_l] = -2i\, \sin(\tfrac{1}{2}k.\Theta l)\, U_{k+l}$.

Let τ be the trace on $C^\infty(\mathbb{T}^n_\Theta)$ defined by $\tau\Big(\sum_{k\in\mathbb{Z}^n} a_k\, U_k\Big) := a_0$ and $\mathcal{H}_\tau$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}^n_\Theta)$ with respect of the norm induced by the scalar product $\langle a, b\rangle := \tau(a^* b)$.

On $\mathcal{H}_\tau = \{\, \sum_{k\in\mathbb{Z}^n} a_k\, U_k \mid \{a_k\}_k \in l^2(\mathbb{Z}^n)\,\}$, we consider the left and right regular representations of $C^\infty(\mathbb{T}^n_\Theta)$ by bounded operators, that we denote respectively by $L(.)$ and $R(.)$.

Let also δ_μ, $\mu \in \{\,1, \dots, n\,\}$, be the n (pairwise commuting) canonical derivations, defined by $\delta_\mu(U_k) := ik_\mu U_k$.

We need to fix notations: let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m+1$ (i.e., $m = \lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$), the square integrable sections of the trivial spin bundle over $\mathbb{T}^n$ and each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$. The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$. The Dirac-like operator is given by

$$\mathcal{D} := -i\,\delta_\mu \otimes \gamma^\mu,$$

where we use hermitian Dirac matrices γ. It is defined and symmetric on the dense subset of $\mathcal{H}$ given by $C^\infty(\mathbb{T}^n_\Theta) \otimes \mathbb{C}^{2^m}$and $\mathcal{D}$ denotes its selfadjoint extension. Thus $C_0\gamma^\alpha = -\varepsilon\gamma^\alpha C_0$, and $\mathcal{D}\ U_k \otimes e_i = k_\mu U_k \otimes \gamma^\mu e_i$, where (e_i) is the canonical basis of $\mathbb{C}^{2^m}$. Moreover, $C_0^2 = \pm 1_{2^m}$ depending on the parity of m. Finally, the chirality in the even case is $\chi := id \otimes (-i)^m \gamma^1 \cdots \gamma^n$. This yields a spectral triple:

Theorem 8.1. *[24, 49] The 5-tuple $(\mathcal{A}_\Theta, \mathcal{H}, \mathcal{D}, J, \chi)$ is a real regular spectral triple of dimension n. It satisfies the finiteness and orientability conditions of Definition 6.2. It is n-summable and its KO-dimension is also n.*

For every unitary $u \in \mathcal{A}$, $uu^* = u^*u = U_0$, the perturbed operator $V_u\,\mathcal{D}\,V_u^*$ by the unitary $V_u := \big(L(u) \otimes 1_{2^m}\big) J \big(L(u) \otimes 1_{2^m}\big) J^{-1}$, must satisfy condition $J\mathcal{D} = \epsilon \mathcal{D} J$. The yields the necessity of a symmetrized covariant Dirac operator $\mathcal{D}_A := \mathcal{D} + A + \epsilon J\,A\,J^{-1}$ since $V_u\,\mathcal{D}\,V_u^* = \mathcal{D}_{L(u)\otimes 1_{2^m}[\mathcal{D}, L(u^*)\otimes 1_{2^m}]}$. Moreover, we get the gauge transformation $V_u \mathcal{D}_A V_u^* = \mathcal{D}_{\gamma_u(A)}$ where the gauged transform one-form of A is $\gamma_u(A) := u[\mathcal{D}, u^*] + uAu^*$, with the shorthand $L(u) \otimes 1_{2^m} \longrightarrow u$. So the spectral action is gauge invariant: $\mathcal{S}(\mathcal{D}_A, f, \Lambda) = \mathcal{S}(\mathcal{D}_{\gamma_u(A)}, f, \Lambda)$.

Any selfadjoint one-form $A \in \Omega^1_{\mathcal{D}}(\mathcal{A})$, is written as $A = L(-iA_\alpha) \otimes \gamma^\alpha,\ A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta$, thus $\mathcal{D}_A = -i\left(\delta_\alpha + L(A_\alpha) - R(A_\alpha)\right) \otimes \gamma^\alpha$. Defining $\tilde{A}_\alpha := L(A_\alpha) - R(A_\alpha)$, we get $\mathcal{D}_A^2 = -g^{\alpha_1\alpha_2}(\delta_{\alpha_1} + \tilde{A}_{\alpha_1})(\delta_{\alpha_2} + \tilde{A}_{\alpha_2}) \otimes 1_{2^m} - \frac{1}{2}\Omega_{\alpha_1\alpha_2} \otimes \gamma^{\alpha_1\alpha_2}$ where

$$\begin{aligned}
\gamma^{\alpha_1\alpha_2} &:= \tfrac{1}{2}(\gamma^{\alpha_1}\gamma^{\alpha_2} - \gamma^{\alpha_2}\gamma^{\alpha_1}),\\
\Omega_{\alpha_1\alpha_2} &:= [\delta_{\alpha_1} + \tilde{A}_{\alpha_1}, \delta_{\alpha_2} + \tilde{A}_{\alpha_2}] = L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2})\\
F_{\alpha_1\alpha_2} &:= \delta_{\alpha_1}(A_{\alpha_2}) - \delta_{\alpha_2}(A_{\alpha_1}) + [A_{\alpha_1}, A_{\alpha_2}].
\end{aligned}$$

8.2 Kernels and dimension spectrum

Since the dimension of the kernel appears in the coefficients (35) of the spectral action, we now compute the kernel of the perturbed Dirac operator:

Proposition 8.2. $\operatorname{Ker}\mathcal{D} = U_0 \otimes \mathbb{C}^{2^m}$, *so* $\dim \operatorname{Ker}\mathcal{D} = 2^m$. *For any selfadjoint one-form A,* $\operatorname{Ker}\mathcal{D} \subseteq \operatorname{Ker}\mathcal{D}_A$ *and or any unitary $u \in \mathcal{A}$,* $\operatorname{Ker}\mathcal{D}_{\gamma_u(A)} = V_u \operatorname{Ker}\mathcal{D}_A$.

One shows that $\operatorname{Ker}\mathcal{D}_A = \operatorname{Ker}\mathcal{D}$ in the following cases:

(i) $A = A_u := L(u) \otimes 1_{2^m}[\mathcal{D}, L(u^*) \otimes 1_{2^m}]$ when u is a unitary in $\mathcal{A}$.

(ii) $||A|| < \frac{1}{2}$.

(iii) The matrix $\frac{1}{2\pi}\Theta$ has only integral coefficients.

Conjecture: $\operatorname{Ker}\mathcal{D} = \operatorname{Ker}\mathcal{D}_A$ *at least for generic Θ's.*

We will use freely the notation (28) about the difference between $\mathcal{D}$ and D.

Since we will have to control small divisors, we give first some Diophantine condition:

Definition 8.3. *Let $\delta > 0$. A vector $a \in \mathbb{R}^n$ is said to be δ-badly approximable if there exists $c > 0$ such that $|q.a - m| \geq c\,|q|^{-\delta}$, $\forall q \in \mathbb{Z}^n \setminus \{\,0\,\}$ and $\forall m \in \mathbb{Z}$.*
We note $\mathcal{BV}(\delta)$ the set of δ-badly approximable vectors and $\mathcal{BV} := \cup_{\delta>0}\mathcal{BV}(\delta)$ the set of badly approximable vectors.

A matrix $\Theta \in \mathcal{M}_n(\mathbb{R})$ (real $n \times n$ matrices) will be said to be badly approximable if there exists $u \in \mathbb{Z}^n$ such that ${}^t\Theta(u)$ is a badly approximable vector of $\mathbb{R}^n$.

Remark. A result from Diophantine approximation asserts that for $\delta > n$, the Lebesgue measure of $\mathbb{R}^n \setminus \mathcal{BV}(\delta)$ is zero (i.e almost any element of $\mathbb{R}^n$ is $\delta-$badly approximable.)

Let $\Theta \in \mathcal{M}_n(\mathbb{R})$. If its row of index i is a badly approximable vector of $\mathbb{R}^n$ (i.e. if $L_i \in \mathcal{BV}$) then ${}^t\Theta(e_i) \in \mathcal{BV}$ and thus Θ is a badly approximable matrix. It follows that almost any matrix of $\mathcal{M}_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$ is badly approximable.

Proposition 8.4. *When $\frac{1}{2\pi}\Theta$ is badly approximable, the spectrum dimension of the spectral triple $\big(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}\big)$ is equal to the set $\{\, n-k \,:\, k \in \mathbb{N}_0 \,\}$ and all these poles are simple. Moreover $\zeta_D(0) = 0$.*

To get this result, we must compute the residues of infinite series of functions on $\mathbb{C}$ and the commutation between residues and series works under the sufficient Diophantine condition.

We can compute $\zeta_D(0)$ easily but the main difficulty is precisely to calculate $\zeta_{D_A}(0)$.

8.3 The spectral action

We fix a self-adjoint one-form A on the noncommutative torus of dimension n.

Proposition 8.5. *If $\frac{1}{2\pi}\Theta$ is badly approximable, then the first elements of the spectral action expansion (33) are given by*
$$⨍\,|D_A|^{-n} = ⨍\,|D|^{-n} = 2^{m+1}\pi^{n/2}\,\Gamma(\tfrac{n}{2})^{-1}, \quad ⨍\,|D_A|^{-n+k} = 0 \text{ for } k \text{ odd}, \quad ⨍\,|D_A|^{-n+2} = 0.$$

Here is the main result of this section.

Theorem 8.6. *Consider the noncommutative torus $\big(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}\big)$ of dimension $n \in \mathbb{N}$ where $\frac{1}{2\pi}\Theta$ is a real $n \times n$ real skew-symmetric badly approximable matrix, and a selfadjoint one-form $A = L(-iA_\alpha) \otimes \gamma^\alpha$. Then, the full spectral action of $\mathcal{D}_A = \mathcal{D} + A + \epsilon JAJ^{-1}$ is*
(i) for $n = 2$, $\mathcal{S}(\mathcal{D}_A, f, \Lambda) = 4\pi\, f_2\,\Lambda^2 + \mathcal{O}(\Lambda^{-2})$,

(ii) for $n = 4$, $\mathcal{S}(\mathcal{D}_A, f, \Lambda) = 8\pi^2\, f_4\,\Lambda^4 - \frac{4\pi^2}{3}\, f(0)\,\tau(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2})$,

(iii) More generally, in $\mathcal{S}(\mathcal{D}_A, f, \Lambda) = \sum_{k=0}^n f_{n-k}\, c_{n-k}(A)\,\Lambda^{n-k} + \mathcal{O}(\Lambda^{-1})$, $c_{n-2}(A) = 0$, $c_{n-k}(A) = 0$ for k odd. In particular, $c_0(A) = 0$ when n is odd.

This result (for $n = 4$) has also been obtained in [41] using the heat kernel method. It is however interesting to get the result via direct computations of (33) since it shows how this formula is efficient.

Remark 8.7. *Note that all terms must be gauge invariants, namely invariant by $A_\alpha \longrightarrow \gamma_u(A_\alpha) = uA_\alpha u^* + u\delta_\alpha(u^*)$. A particular case is $u = U_k$ where $U_k\delta_\alpha(U_k^*) = -ik_\alpha U_0$.*

In the same way, note that there is no contradiction with the commutative case where, for any selfadjoint one-form A, $\mathcal{D}_A = \mathcal{D}$ (so A is equivalent to 0!), since we assume in Theorem 8.6 that Θ is badly approximable, so $\mathcal{A}$ cannot be commutative.

Conjecture 8.8. *The constant term of the spectral action of $\mathcal{D}_A$ on the noncommutative n-torus is proportional to the constant term of the spectral action of $\mathcal{D} + A$ on the commutative n-torus.*

Remark 8.9. *The appearance of a Diophantine condition for Θ has been characterized in dimension 2 by Connes [22, Prop. 49] where in this case, $\Theta = \theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $\theta \in \mathbb{R}$. In fact, the Hochschild cohomology $H(\mathcal{A}_\Theta, \mathcal{A}_\Theta{}^*)$ satisfies dim $H^j(\mathcal{A}_\Theta, \mathcal{A}_\Theta{}^*) = 2$ (or 1) for $j = 1$ (or $j = 2$) if and only if the irrational number θ satisfies a Diophantine condition like $|1 - e^{i2\pi n\theta}|^{-1} = \mathcal{O}(n^k)$ for some k.*
The result of Theorem 8.6 without this Diophantine condition is unknown.
Recall that when the matrix Θ is quite irrational (the lattice generated by its columns is dense after translation by $\mathbb{Z}^n$, see [49, Def. 12.8]), then the C^*-algebra generated by $\mathcal{A}_\Theta$ is simple.
It is possible to go beyond the Diophantine condition: see [41].

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References

[1] W. Arendt, R. Nittka, W. Peter and F. Steiner, "Weyl's law: Spectral properties of the Laplacian in mathematics and physics", in *Mathematical Analysis of Evolution, Information, and Complexity*, W. Arendt and W. Schleich ed., Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim, 2009.

[2] C. Bär, "Conformal structures in noncommutative geometry", J. Noncomm. Geom. **1** (2007), 385–395.

[3] R. Beals and P. Greiner, *Calculus on Heisenberg Manifolds*, Princeton University Press, Princeton, 1988.

[4] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag, Berlin Heidelberg New York, 1992.

[5] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin Heidelberg New York, 2002.

[6] [J.-P. Bourguignon and P. Goduchon, "Spineurs, opérateurs de Dirac et variations", Commun. Math. Phys. **144** (1992), 581–599.

[7] R. Brooks and C. Gordon, "Isospectral families of conformally equivalent Riemannian metric", Bull. Amer. Math. Soc. **23** (1990), 433–436.

[8] A. Carey, V. Gayral, A. Rennie and F. Sukochev, "Integration on locally compact noncommutative spaces", J. Funct. Anal. **263** (2012), 383–414.

[9] A. Carey, J. Phillips and F. Sukochev, "Spectral flow and Dixmier traces", Adv. Math. **173** (2003), 68–113.

[10] A. Chamseddine and A. Connes, "The spectral action principle", Commun. Math. Phys. **186** (1997), 731–750.

[11] A. Chamseddine and A. Connes, "Universal formula for noncommutative geometry actions: Unification of gravity and the standard model", Phys. Rev. Letters **77** (1996), 4868–4871.

[12] A. Chamseddine and A. Connes, "Inner fluctuations of the spectral action", J. Geom. Phys. **57** (2006), 1–21.

[13] A. Chamseddine and A. Connes, "Scale invariance in the spectral action", J. Math. Phys. **47** (2006), 063504.

[14] A. Chamseddine and A. Connes, "Quantum gravity boundary terms from the spectral action on noncommutative space", Phys. Rev. Lett. **99** (2007), 071302.

[15] A. Chamseddine and A. Connes, "Why the Standard Model", J. Geom. Phys. **58** (2008), 38–47.

[16] A. Chamseddine and A. Connes, "The uncanny precision of the spectral action", Commun. Math. Phys. **293** (2010), 867–897.

[17] A. Chamseddine and A. Connes, "Noncommutative geometry as a framework for unification of all fundamental interactions including gravity. Part I", Fortschritte der Physik **58** (2010), 553–600.

[18] A. Chamseddine and A. Connes, "Noncommutative geometric spaces with boundary: spectral action", J. Geom. Phys. **61** (2011), 317–332.

[19] A. Chamseddine and A. Connes "Spectral Action for Robertson–Walker metrics", arXiv:1105.4637v1 [hep-th].

[20] A. Chamseddine, A. Connes and M. Marcolli, "Gravity and the standard model with neutrino mixing", Adv. Theor. Math. Phys. **11** (2007), 991–1090.

[21] A. Connes, "C^*-algèbres et géométrie différentielle", C. R. Acad. Sci. Paris **290** (1980), 599–604.

[22] A. Connes, "Noncommutative differential geometry", Pub. Math. IHÉS **39** (1985), 257–360.

[23] A. Connes, "The action functional in non-commutative geometry", Commun. Math. Phys. **117** (1988), 673–683.

[24] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[25] A. Connes, "Noncommutative geometry and reality", J. Math. Phys. **36** (1995), 6194–6231.

[26] A. Connes, "Gravity coupled with matter and the foundation of non commutative geometry", Commun. Math. Phys. **182** (1996), 155–177.

[27] A. Connes, "Noncommutative geometry and the standard model with neutrino mixing", JHEP **0611**:081 (2006).

[28] A. Connes, "On the spectral characterization of manifolds", arXiv:0810.2088v1.

[29] A. Connes, "A unitary invariant in Riemannian geometry", Inter. Journ. Geom. Meth. Mod. Phys. **5** (2008), 1215–1242.

[30] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Colloquium Publications, Vol. 55, American Mathematical Society, 2008.

[31] A. Connes and H. Moscovici, "Transgression du caractère de Chern et cohomologie cyclique", C. R. Acad. Sci. Paris, Série **I 303** (1986), 913–918.

[32] A. Connes and H. Moscovici, "Transgression and Chern character of finite-dimensional K-cycles", Commun. Math. Phys. **155** (1993), 103–122.

[33] A. Connes and H. Moscovici, "The local index formula in noncommutative geometry", Geom. Funct. Anal. **5** (1995), 174–243.

[34] J. Dixmier, "Existence de traces non normales", C. R. Acad. Sci. Paris **262A** (1966), 1107–1108.

[35] A. Edery, "Multidimensional cut-off technique, odd-dimensional Epstein zeta functions and Casimir energy of massless scalar fields", J. Phys. A: Math. Gen. **39** (2006), 678–712.

[36] D. Essouabri, B. Iochum, C. Levy and A. Sitarz, "Spectral action on noncommutative torus", J. Noncommut. Geom. **2** (2008), 53–123.

[37] R. Estrada, J. M. Gracia-Bondía and J. C. Várilly, "On summability of distributions and spectral geometry", Commun. Math. Phys. **191** (1998), 219–248.

[38] S. Fulling, "Some properties of Riesz means and spectral expansions", Electron. J. Diff. Equ. **6**(1999), 1–39.

[39] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, "Moyal planes are spectral triples", Commun. Math. Phys. **246** (2004), 569–623.

[40] V. Gayral and B. Iochum, "The spectral action for Moyal plane", J. Math. Phys. **46** (2005), 043503.

[41] V. Gayral, B. Iochum and D. Vassilevich, "Heat kernel and number theory on NC-torus", Commun. Math. Phys. **273** (2007), 415–443.

[42] S. Giddings, D. Marlof and J. Hartle, “Observables in effective gravity”, Phys. Rev. D **74** (2006) 064018.

[43] P. B. Gilkey, *Invariance Theory, the Heat equation, and the Atiyah–Singer Index Theory*, CRC Press, Boca Raton, 1995.

[44] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry*, Chapman & Hall, 2003.

[45] P. B. Gilkey and G. Grubb, “Logarithmic terms in asymptotic expansions of heat operator traces”, Comm. Partial Diff. Equ. **23** (1998), 777–792.

[46] N. Ginoux, *The Dirac spectrum*, Lecture Notes in Mathematics 1976, Springer-Verlag, Berlin Heidelberg 2009.

[47] G. Grubb and R. Seeley, “Weakly parametric pseudodifferential operators and the Atiyah–Patodi–Singer boundary problem”, Invent. Math. **121** (1995), 481–529.

[48] J. M. Gracia-Bondía and J. C. Várilly, “Algebras of distributions suitable for phase-space quantum mechanics I”, J. Math. Phys. **29** (1988), 869–879.

[49] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, 2001.

[50] H. Grosse and R. Wulkenhaar, “8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory”, J. Geom. Phys. **62** (2012), 1583–1599.

[51] V. Guillemin, “A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues”, Adv. Math. **55** (1985), 131–160.

[52] V.W. Guillemin, S. Sternberg and J. Weitsman, “The Ehrhart function for symbols”, arXiv:math.CO/06011714.

[53] F. Hanisch, F. Pfäffle and C. Stephan, “The spectral action for Dirac operators with skew-symmetric torsion”, Commun. Math. Phys. **300** (2010, 877–888.

[54] G. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[55] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon, Oxford, 1979.

[56] O. Hijazi, “Spectral properties of the Dirac operator and geometrical structures”, Lecture notes for the Summer School in Villa de Leyva, Columbia, 1999. `http://www.iecn.u-nancy.fr/ hijazi/HTML/l2.pdf`

[57] B. Iochum, D. Kastler and T. Schücker, “Fuzzy mass relations for the Higgs”, J. Math. Phys. **36** (1995), 6232–6254.

[58] B. Iochum and C. Levy, “Tadpoles and commutative spectral triples”, J. Noncommut. Geom. **5** (2011), 299–329.

[59] B. Iochum and C. Levy, “Spectral triples and manifolds with boundary”, J. Func. Anal. **260** (2011), 117–134.

[60] B. Iochum, C. Levy and A. Sitarz, “Spectral action on $SU_q(2)$”, Commun. Math. Phys. **289** (2009), 107–155.

[61] B. Iochum, C. Levy and D. Vassilevich, “Spectral action for torsion with and without boundaries”, Commun. Math. Phys., **310** (2012), 367–382.

[62] B. Iochum, C. Levy and D. Vassilevich, “Global and local aspects of spectral actions”, J. Phys. A: Math. Theor. **45** (2012), 374020.

[63] J.-H. Jureit and C. Stephan, “On a classification of irreducible almost commutative geometries IV”, J. Math. Phys. **49** (2008), 033502.

[64] J.-H. Jureit and C. Stephan, “Finding the standard model of particle physics, a combinatorial problem”, Computer Phys. Commun. **178** (2008), 230–247.

[65] T. Kato, *Perturbation Theory For Linear Operators*, Springer–Verlag, Berlin-Heidelberg-New York, 1980.

[66] D. Kolodrubetz and M. Marcolli, “Boundary conditions of the RGE flow in the noncommutative geometry approach to particle physics and cosmology”, Phys. Lett. **B 693** (2010), 166–174.

[67] T. Krajewski, “Classification of finite spectral triples”, J. Geom. Phys. **18** (1997), 1–30.

[68] G. Landi, *An Introduction to Noncommutative Spaces and Their Geometries*, Springer-Verlag, Berlin Heidelberg 1997.

[69] G. Landi and C. Rovelli, “General relativity in terms of Dirac eigenvalues”, Phys. Rev. Lett. **78** (1997), 3051–3054.

[70] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, 1989.

[71] F. Lizzi, G. Mangano, G. Miele and G. Sparano, “Constraints on unified gauge theories from noncommutative geometry”, Mod. Phys. Lett. **A11** (1996), 2561–2572.

[72] S. Lord, D. Potapov and F. Sukochev, “Measures from Dixmier traces and zeta functions”, J. Funct. Anal. **259** (2010), 1915–1949.

[73] S. Lord, A. Sedaev and F. Sukochev, “Dixmier traces as singular symmetric functionals and applications to measurable operators”, J. Funct. Anal. **224** (2005), 72–106.

[74] S. Lord and F. Sukochev, “Noncommutative residues and a characterization of the noncommutative integral”, Proc. Amer. Math. Soc. **139** (2011), 243–257.

[75] S. Lord and F. Sukochev, “Measure Theory in Noncommutative Spaces”, in “Symmetry, Integrability and Geometry: Methods and Applications”, SIGMA **6** (2010), 072, 36 pages.

[76] M. Marcolli, E. Pierpaoli and K. Teh, “The spectral action and cosmic topology”, Commun. Math. Phys. **304** (2011), 125–174.

[77] M. Marcolli, E. Pierpaoli and K. Teh, "The coupling of topology and inflation in noncommutative cosmology", Commun. Math. Phys. **309** (2012), 341–360.

[78] M. Marcolli and E. Pierpaoli, "Early universe models from noncommutative geometry", Adv. Theor. Math. Phys. **14** (2010), 1373–1432.

[79] C. P. Martin, J. M. Gracia-Bondía and J. C. Várilly, "The standard model as a noncommutative geometry: the low-regime", Phys. Rep. **294** (1998), 363–406.

[80] J. E. Moyal, "Quantum mechanics as a statistical theory", Proc. Camb. Philos. Soc. **45** (1949), 99–124.

[81] W. Nelson and M. Sakellariadou, "Inflation mechanism in asymptotic noncommutative geometry ", Phys. Lett. **B 680** (2009), 263–266.

[82] F. Nicola and L. Rodino, *Global Pseudodifferential Calculus on Euclidean Spaces* Birkäuser, Springer Basel, 2010.

[83] S. Paycha, "Regularized integrals, sums and traces. An analytic point of view (Preliminary draft)", 2010.

[84] F. Pfäffle and C. Stephan, "On gravity, torsion and the spectral action principle", J. Funct. Anal. **261** (2012), 1529–1565).

[85] F. Pfäffle and C. Stephan, "The Holst action by the spectral action principle", Commun. Math. Phys. **307** (2011), 261–273.

[86] R. Ponge, "Schwartz kernels and noncommutative residue (Preliminary draft)", `R. Ponge's homepage`

[87] R. Ponge, "Calcul hypoelliptique sur les variétés de Heisenberg, résidu non commutatif et géométrie pseudo-hermitienne", PhD Thesis, Univ. Paris-Sud (Orsay), Dec. 2000, `http://www.ms.u-tokyo.ac.jp/ ponge/theses/these.pdf`

[88] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[89] A. Rennie, "Commutative geometries are spin manifolds", Rev. Math. Phys. **13** (2001), 409–464.

[90] A. Rennie, "Poincaré duality and spin^c structures for complete noncommutative manifolds", math-ph/0107013.

[91] A. Rennie, "Smoothness and locality for nonunital spectral triples", K-Theory, **28** (2003), 127–165.

[92] A. Rennie and J. Várilly, "Reconstruction of manifolds in noncommutative geometry", arXiv:math/0610418, (2006).

[93] M. A. Rieffel, "C^*-algebras associated with irrational rotations", Pac. J. Math. **93** (1981), 415–429.

[94] M. A. Rieffel, *Deformation Quantization for Actions of* $\mathbb{R}^d$, Memoirs of the Amer. Math. Soc. **506**, Providence, RI, 1993.

[95] M. Sakellariadou, "Cosmology within noncommutative spectral geometry", arXiv:1101.2174v1 [hep-th].

[96] S. Schopka, "Noncommutative Einstein manifolds", Shaker Verlag GmbH, Germany, 2007.

[97] T. Schücker, "Forces from Connes' geometry", in 'Topology and Geometry in Physics', Eds: E. Bick and F. Steffen Lecture Notes in Physics **659** (2005), 285–350.

[98] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris, 1979.

[99] M. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer Verlag 1987, 2nd ed. 2001.

[100] B. Simon, *Trace Ideals and Their Applications*, Second Edition, AMS Mathematical Surveys and Monographs 120, Providence, 2005.

[101] M. Takesaki, *Theory of Operator Algebras II*, Springer-Verlag, Berlin Heidelberg New York 2003.

[102] M. Taylor, *Pseudodifferential Operators*, Princeton Univ. Press, Princeton, NJ, 1981.

[103] F. J. Vanhecke, "On the product of real spectral triples", Lett. Math. Phys. **50** (1999), 151–162.

[104] J. Várilly, *Dirac Operators and Spectral Geometry*, `http://www.impan.pl/Manuals/Varilly_dosg.pdf`

[105] J. Várilly, *An Introduction to Noncommutative Geometry*, European Math. Soc. Publishing House (EMS Series of Lectures in Mathematics), 2006.

[106] J. C. Várilly and J. M. Gracia-Bondía, "Algebras of distributions suitable for phase-space quantum mechanics II: Topologies on the Moyal algebra", J. Math. Phys. **29** (1988), 880–887.

[107] D. Vassilevich, "Heat kernel expansion: user's manual", Phys. Rep. **388** (2003), 279–360.

[108] A. van de Ven, "Index-free heat kernel coefficients", Class. Quant. Grav. **15** (1998), 2311–2344.

[109] M. Wodzicki, "Local invariants of spectral asymmetry", Invent. Math. **75** (1984), 143–177.

[110] M. Wodzicki, "Noncommutative residue. Chapter I: Fundamentals", 320–399, in *K-theory, Arithmetic and Geometry*, Yu. I. Manin, ed., Lecture Notes in Math. **1289**, Springer, Berlin 1987.